Hyperbolic structures on a toric arrangement complement

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Hyperbolic structures on a toric arrangement complement

Hyperbolische structuren op het complement van een torische schikking

(met een samenvatting in het Nederlands)

Proefschrift

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dedicated to the memory of my father

Contents

Introduction

This thesis could be viewed as a mergence of two lines of research. One line is the study on the hypergeometric functions associated with a root system. This is actually a multivariable analogue of the classical Euler-Gauss hypergeometric functions. These functions were introduced and studied by Heckman and Opdam in a series of papers [**16, 17, 29, 30**] in the eighties and nineties of the last century. The other one is the work by Couwenberg, Heckman and Looijenga (2005) in [**6**], which studies the geometric structures on the projective arrangement complements. This work actually includes the theory of Deligne-Mostow (1986) [**11**] on the Lauricella functions as a special case, which provides ball quotient structures on $\mathbb{P}^n$ minus a hyperplane configuration of type A_{n+1}. This thesis adopts Couwenberg-Heckman-Looijenga's point of view to investigate the root system hypergeometric functions and hence studies the hyperbolic structure on the corresponding toric arrangement complements. This is why we think of this thesis as a mergence of the aforementioned two lines of research.

We first construct a projective structure on a toric arrangement complement. The basic idea is that we can write a projective structure on a complex manifold M in terms of an affine structure on $M \times \mathbb{C}^\times$. It is well-known that an affine structure on a complex manifold is given by a torsion free and flat connection on its (co)tangent bundle, and vice versa. So constructing a projective structure on M is equivalent to producing a torsion free and flat connection on $M \times \mathbb{C}^\times$. We start with an adjoint torus $H := \mathrm{Hom}(Q, \mathbb{C}^\times)$ given by a root lattice $Q := \mathbb{Z}R$ where R is a reduced irreducible root system. Denote the Lie algebra of H by $\mathfrak{h}$ and the Weyl group of R by W. We are also given a toric arrangement associated with a root system R, that is, a finite collection of hypertori each of which is defined by $H_\alpha := \{h \in H \mid e^\alpha(h) = 1\}$ where e^α is a character of H. We write H° for the complement of the union of these hypertori. Let κ be a W-invariant multiplicity parameter for R defined by $\kappa := (k_\alpha)_{\alpha \in R} \in \mathbb{C}^R$. Inspired by the special hypergeometric system constructed by Heckman and Opdam, we consider for $u, v \in \mathfrak{h}$, such a second order differential operator on $\mathcal{O}_{H^\circ}$:

$$D^\kappa_{u,v} := \partial_u\partial_v + \frac{1}{2}\sum_{\alpha>0} k_\alpha \alpha(u)\alpha(v)\frac{e^\alpha+1}{e^\alpha-1}\partial_{\alpha^\vee} + \partial_{b^\kappa(u,v)} + a^\kappa(u,v)$$

where ∂_u denotes the associated translation invariant vector field on H for any $u \in \mathfrak{h}$ and

$$a^\kappa : \mathfrak{h}\times\mathfrak{h} \to \mathbb{C}, \quad b^\kappa : \mathfrak{h}\times\mathfrak{h}\to\mathfrak{h}$$

are a W-invariant bilinear form and a W-equivariant bilinear map respectively. We want this system to define a projective structure on H°. That means for each multiplicity parameter κ and each W-equivariant bilinear map b^κ, there exists a W-invariant bilinear form a^κ such that the system of differential equations $D^\kappa_{u,v}f = 0$ for all $u, v \in \mathfrak{h}$ is integrable. In order to see the integrability of the system, we treat it from a different point of view, i.e., the one from the work of Couwenberg-Heckman-Looijenga. Now we associate to these data connections $\nabla^\kappa = \nabla^0 + \Omega^\kappa$ and $\tilde\nabla^\kappa = \tilde\nabla^0 + \tilde\Omega^\kappa$ on the cotangent bundles of H° and $H^\circ \times \mathbb{C}^\times$ with $\Omega^\kappa \in \mathrm{Hom}(\Omega_{H^\circ}, \Omega_{H^\circ}\otimes\Omega_{H^\circ})$ given by

$$\Omega^\kappa : \zeta \in \Omega_{H^\circ} \mapsto \frac{1}{2}\sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1}\zeta(\partial_{\alpha^\vee})d\alpha\otimes d\alpha + (B^\kappa)^*(\zeta)$$

and $\tilde\Omega^\kappa \in \mathrm{Hom}(\Omega_{H^\circ\times\mathbb{C}^\times}, \Omega_{H^\circ\times\mathbb{C}^\times}\otimes\Omega_{H^\circ\times\mathbb{C}^\times})$ given by

$$\tilde\Omega^\kappa : \begin{cases} \zeta \in \Omega_{H^\circ} \mapsto \frac{1}{2}\sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1}\zeta(\partial_{\alpha^\vee})d\alpha\otimes d\alpha + (B^\kappa)^*(\zeta) \\ \qquad\qquad - \zeta\otimes\frac{dt}{t} - \frac{dt}{t}\otimes\zeta, \\ \frac{dt}{t} \in \Omega_{\mathbb{C}^\times} \mapsto A^\kappa - \frac{dt}{t}\otimes\frac{dt}{t}. \end{cases}$$

Here ∇^0 and $\tilde\nabla^0$ denote the (flat) translation invariant connections on H and $H\times\mathbb{C}^\times$ respectively, t is the coordinate for $\mathbb{C}^\times$, and A^κ and B^κ denote the translation invariant tensor fields on H or $H\times\mathbb{C}^\times$ defined by a^κ and b^κ respectively. We can show that the system defined by $D^\kappa_{u,v}f = 0$ for all $u, v\in\mathfrak{h}$ is integrable if and only if the connection $\tilde\nabla^\kappa$ given above defines an affine structure, i.e., the connection $\tilde\nabla^\kappa$ is torsion free and flat. The torsion freeness of $\tilde\nabla^\kappa$ comes directly from the torsion freeness of ∇^κ while the flatness of $\tilde\nabla^\kappa$ needs more effort. In order to check the flatness of $\tilde\nabla^\kappa$, we need to invoke a flatness criterion set up by Looijenga [**21**], or by Kohno [**19**] at an earlier time. This criterion requires us to compactify $H^\circ\times\mathbb{C}^\times$ and compute the residues of $\tilde\Omega^\kappa$ along those added mirrors and boundary divisors. Then by applying the criterion to our situation, we can obtain the conditions for $\tilde\nabla^\kappa$ being flat. According to these conditions, we can find an appropriate bilinear form a^κ so that the connection $\tilde\nabla^\kappa$ is indeed flat and hence a W-invariant projective structure is constructed on H° in terms of ∇^κ. This work is presented in Chapter 2.

We next show that the toric arrangement complement H° admits a hyperbolic structure when κ lies in some certain region so that its image under the projective evaluation map lands in a complex ball. The basic idea is that we first identify the monodromy representation of the system with the reflection representation and thus define a Hermitian form h on the image of the evaluation map for each κ, then we can find its hyperbolic region by computing its determinant and show that its dual Hermitian form h^* is greater than 0 (equivalently $h < 0$) so that the desired result follows. We first compute the eigenvalues of the residue endomorphisms of $\tilde{\nabla}^\kappa$ along mirrors and boundary divisors respectively and a surprising fact is that there are at most two eigenvalues for each residue endomorphism no matter whether along a mirror or a boundary divisor. This actually tells us what the local behavior of the evaluation map looks like for the affine structure around those divisors. Then we construct the reflection representation of the so-called affine Artin group $\mathrm{Art}(M)$ where M is the affine Coxeter matrix associated with the affine root system $\tilde{R}$ of R, while the extended affine Artin group $\mathrm{Art}'(M)$ $(:= \mathrm{Art}(M) \rtimes (P^\vee/Q^\vee))$ can be identified with the fundamental group of the orbifold $W\backslash H^\circ$ by Brieskorn's theorem, hence we can identify the reflection representation with the monodromy representation of the system accordingly. We further define a Hermitian form h on the corresponding target space A from the point of view of the reflection representation so that we can obtain the hyperbolic region of the system by investigating its determinant. For our situation we can write out the evaluation map around those subregular points in the form of local coordinates in terms of those local exponents. Here by subregular points we mean those points lying in one and only one mirror or boundary divisor. Prepared by these, finally we can prove the dual Hermitian form h^* is greater than zero when κ lies in the hyperbolic region so that the Γ-covering of $W\backslash H^\circ$ admits a complex ball structure, where Γ stands for the projective monodromy group. This work is done in Chapter 3.

The original goal of this PhD research project is to show that $W\backslash H^\circ$ can be biholomorphically mapped onto a Heegner divisor complement of a ball quotient $\Gamma\backslash\mathbb{B}$ by a projective evaluation map if the so-called Schwarz conditions are satisfied. But unfortunately we haven't gone that far in this thesis, we hope we can address this issue in a following paper. In fact, our situation for the case of type A_n corresponds to the Deligne-Mostow theory which provides ball quotient structures on $\mathbb{P}^n$ as well as their modular interpretations. Besides, Looijenga also showed in his paper [**24**] that the orbifold $SW\backslash H^\circ(E_n)$ $(n = 6, 7, 8)$ is isomorphic to the moduli space of the Del Pezzo triples (S, K, p) of degree $d := 9 - n$ for which $SW := \{\pm 1\}.W$ and K is an irreducible rational curve with a simple node at p. We denote this moduli space by $\tilde{\mathcal{M}}(d)$. Forgetting about (K, p), we get a moduli space of Del Pezzo surfaces of degree d, denoted by $\mathcal{M}(d)$. While Allcock, Carlson and Toledo [**1**], Kondo [**20**], and Heckman and Looijenga [**14**] respectively showed that the moduli space of Del Pezzo surfaces of degree d has a ball quotient structure under a period map for $d = 3, 2, 1$, all from a viewpoint of Hodge theory. So inspired by these cases, we wish to find, for each root system, a suitable projective evaluation map in our situation

$$Pev : W\backslash H^\circ \hookrightarrow \Gamma\backslash\mathbb{B}$$

so that $W\backslash H^\circ$ is isomorphic onto a Heegner divisor complement and in particular, for type E_n $(n = 6, 7, 8)$, the following diagram commutes.

$$\begin{array}{ccc}
\tilde{\mathcal{M}}(d) & \xrightarrow{\text{forget } (K,p)} & \mathcal{M}(d) \\
\simeq\Big\downarrow & & \Big\downarrow Per \\
SW\backslash H^\circ(E_n) & \xrightarrow{Pev} & \Gamma\backslash\mathbb{B}
\end{array}$$

So a much more ambitious goal is to give each ball quotient obtained in this way a modular interpretation, but we have to say we barely have any clue about this for the moment except for only limited few cases already mentioned above.

As a byproduct of this research, we construct a Frobenius algebra structure on $H^\circ \times \mathbb{C}^\times$. Since it is an integrable system, we would speculate there exists a "Frobenius-type" structure on it. In fact, we can construct such a Frobenius algebra on $H^\circ \times \mathbb{C}^\times$ with the trace map given by the symmetric bilinear map a^κ. We believe we can go further in this direction, e.g., to find a potential function for the structure, although we haven't arrived there in this thesis due to the limited time. This quite preliminary work is presented in Chapter 4.

For the completeness of this thesis, we review some beautiful theories which directly motivated our current research. They are the Deligne-Mostow theory on the Lauricella functions, Couwenberg-Heckman-Looijenga's theory of geometric structures on projective arrangement complements and the theory of torus embeddings respectively. This is done in Chapter 1.

CHAPTER 1

Preliminary theories

In this chapter we review some beautiful theories which directly motivated the research presented in this thesis. In Section 1.1, we briefly review the Deligne-Mostow theory which studies the geometry and monodromy around the Lauricella F_D functions. In Section 1.2, we give a short overview on the Couwenberg-Heckman-Looijenga's theory of geometric structure on projective arrangement complements and thus extends the Deligne-Mostow theory to a more general setting. In Section 1.3, we briefly introduce the torus embeddings theory since we are dealing with geometric structure on arrangement complements for the toric situation in this thesis.

1.1. The Deligne-Mostow theory of Lauricella functions

The study on the hypergeometric functions has a long history which can be traced back to Euler in the 18^{th} century. Then it started to attract more and more attention due to the impressive results obtained by Gauss on the *hypergeometric function* that is now named after him. Later Riemann developed a nice way to study the monodromy associated to the Gauss function and Schwarz specified all those parameters for its monodromy group being finite. Klein generalized this work to determine whether the monodromy group is discrete.

Subsequently, this classical work was generalized to hypergeometric functions in several variables. Its two variables' version was introduced by Appell and the corresponding monodromy problem was studied by Picard. A little later, Lauricella generalized this to the functions in arbitrarily many variables which are called F_D, F_A, F_B and F_C. Terada made some progress on the monodromy problem for the Lauricella functions. Then in 1986 Deligne and Mostow published their famous paper [**11**] which completed the work of Picard and Terada on the (Appell-)Lauricella functions. They gave a rigorous treatment of the geometry and monodromy problem associated to these functions and discovered ball quotient structures on $\mathbb{P}^n$ minus the hyperplane configuration of type A_{n+1}. Meanwhile Thurston in his paper [**33**] used a combinatorial method and the theory of cone manifolds to study a related moduli problem and provided another point of view.

This section is intended to give a short review on the Deligne-Mostow theory of Lauricella F_D functions. Besides the original paper, good expositions can be found in [**23**] and in Chapter 2 of Couwenberg's PhD thesis [**5**]. We shall use [**23**] as main source in this section, but for the sake of brevity, we leave out most of the proofs.

Let be given real numbers $\mu_0, \cdots, \mu_n$ in the interval $(0,1)$, where $n \in \mathbb{Z}_+$. This $(n+1)$-tuple $\mu := (\mu_0, \cdots, \mu_n)$ is referred to as a *weight system* and we call its sum $|\mu| := \sum_{i=0}^n \mu_i$ the *total weight* of μ. Then the *Lauricella differential* of weight μ is given by

$$\psi_z := (z_0 - \xi)^{-\mu_0} \cdots (z_n - \xi)^{-\mu_n} d\xi$$

where $z = (z_0, \cdots, z_n) \in (\mathbb{C}^{n+1})^\circ$. Here we denote by $(\mathbb{C}^{n+1})^\circ$ the set of $z = (z_0, \cdots, z_n) \in \mathbb{C}^{n+1}$ whose components are pairwise distinct. This could also be regarded as the configuration space of $n+1$ distinct ordered points in $\mathbb{C}$. Although this is a multivalued differential, it has a natural branch on a left half plane by taking there the value of $(-\xi)^{-|\mu|}$ whose argument lies in $(-\pi/|\mu|, \pi/|\mu|)$. Near z_k, we notice that ψ_z is of the form $(\xi - z_k)^{-\mu_k} \exp(holom) d\xi$ and it is the differential of a function of the form $const + (\xi - z_k)^{1-\mu_k} \exp(holom)$. Since $1 - \mu_k > 0$, that function takes a well-defined value in z_k. This implies that ψ_z can be integrated along every *relative arc* of $(\mathbb{C}, \{z_0, \cdots, z_n\})$; here we call an oriented piecewise differentiable arc in $\mathbb{C}$ whose end points lie in $\{z_0, \cdots, z_n\}$ a relative arc of $(\mathbb{C}, \{z_0, \cdots, z_n\})$.

It is natural to study the behavior of this differential at infinity by taking the substitution $\xi = \omega^{-1}$ and seeing what happens at $\omega = 0$, then we have

$$\psi_z = -(\omega z_0 - 1)^{-\mu_0} \cdots (\omega z_n - 1)^{-\mu_n} \omega^{-(2-|\mu|)} d\omega,$$

which suggests to put $z_{n+1} := \infty$ and $\mu_{n+1} := 2 - |\mu|$. It is also integrable at z_{n+1} in case $\mu_{n+1} < 1$.

Let be given a relative arc γ_{z° of $(\mathbb{C}, \{z_0^\circ, \cdots, z_n^\circ\})$ and a branch of ψ_{z° on γ_{z° so that $\int_{\gamma_{z^\circ}} \psi_{z^\circ}$ is defined. Choose open disks D_k about z_k° in $\mathbb{C}$ such that these $D_0, \cdots, D_n$ do not intersect each other. Then we call such a function

$$f : D_0 \times \cdots \times D_n \to \mathbb{C}$$
$$z \mapsto \int_{\gamma_z} \psi_z$$

a *Lauricella function* of weight μ. The function is well-defined since we can find for every $z \in D_0 \times \cdots \times D_n$, a relative arc γ_z of $(\mathbb{C}, \{z_0, \cdots, z_n\})$ and a branch of ψ_z on γ_z such that both depend continuously on z and yield the prescribed value when $z = z^\circ$. This function is also holomorphic since any primitive of ψ near (z°, z_k°) is of the form $g(z) + (\xi - z_k)^{1-\mu_k} h(\xi, z)$ with g and h being holomorphic as previously described.

We note that the Lauricella functions define a local system of $\mathbb{C}$-vector spaces: its stalk L_z at z is the space of germs of holomorphic functions at $z \in (\mathbb{C}^{n+1})^\circ$ that are in fact germs of Lauricella functions and we can naturally identify L_z with L_{z° (by the analytic continuation) for any $z \in D_0 \times \cdots \times D_n$.

With little effort, we obtain the elementary properties of Lauricella functions.

Proposition 1.1. *Any* $f \in L_z$

(i) is translation invariant: $f(z_0 + a, \cdots, z_n + a) = f(z_0, \cdots, z_n)$ *for small* $a \in \mathbb{C}$,

(ii) is homogeneous of degree $1-|\mu|$: $f(e^t z_0, \cdots, e^t z_n) = e^{(1-|\mu|)t} f(z_0, \cdots, z_n)$ *for small* $t \in \mathbb{C}$ *and*

(iii) obeys the system of differential equations

$$\frac{\partial^2 f}{\partial z_k \partial z_l} = \frac{1}{z_k - z_l}(\mu_l \frac{\partial f}{\partial z_k} - \mu_k \frac{\partial f}{\partial z_l}), \quad 0 \leq k < l \leq n. \tag{1.1}$$

The translation invariance of the Lauricella functions implies that they are in fact locally defined on $V_n^\circ := (\mathbb{C}^{n+1})^\circ/\text{main diagonal}$. The homogeneity implies that when $|\mu| = 1$, these functions are constant on the $\mathbb{C}^\times$-orbits and hence define a local system on $\mathbb{P}(V_n^\circ)$; we call this the *parabolic* case. Satisfying such a system of differential equations shows that the map assigning to $f \in L_z$ its 1-jet at z is injective, which can be seen by letting $f_k := \frac{\partial f}{\partial z_k}$ and rewriting the above equations (1.1) to the system of ODE's

$$\frac{\partial f_k}{\partial z_l} = \frac{1}{z_k - z_l}(\mu_l f_k - \mu_k f_l), \quad k \neq l.$$

Then we can define an *L-slit* to be an oriented piecewise differentiable arc in $\mathbb{P}^1$ by adding to a relative arc of $(\mathbb{C}, \{z_0, \cdots, z_n\})$ a line from z_n to $z_{n+1} := \infty$ which is eventually parallel to the real axis in the positive direction. We denote by δ such an L-slit or equivalently the system of arcs $(\delta_1, \cdots, \delta_{n+1})$ if we denote the piece connecting z_{k-1} with z_k by δ_k. As a counterpart to δ_k, we denote by δ_k^- the arc connecting z_{k-1} with z_k that is 'infinitesimally' close to δ_k approached from the right side. Then the value of ψ_z on δ_k^- is given as taking its limit from the right. We have such a relation between the value of ψ_z on δ_k and δ_k^-:

$$\psi_z|\delta_k^- = \exp(-2\pi\sqrt{-1}(\mu_0 + \cdots + \mu_{k-1})).\psi_z|\delta_k.$$

We have the following important theorem.

Theorem 1.2. *The functions $\int_{\delta_1}\psi, \cdots, \int_{\delta_n}\psi$ form a basis for L_z. Moreover, L_z contains the constant functions if and only if we are in the parabolic case: $|\mu| = 1$.*

The idea of the proof is to choose a closed piecewise differentiable path γ in $\mathbb{C}$ that is the boundary of an embedded disk $D \subset \mathbb{C}$ whose interior contains no z_k's, and to choose a branch of ψ_z over D so that we have $\int_\gamma \psi_z = 0$. Then $\int_\gamma \psi_z$ is a sum of Lauricella functions associated to simple relative arcs. When $|\mu| = 1$, ψ_z is equal to $\omega^{-1}d\omega$ near ∞. So then for a loop γ which encircles $z_0, \cdots, z_n$ in the clockwise direction, we have $\int_\gamma \psi_z = \int_{\omega(\gamma)} \omega^{-1}d\omega = 2\pi\sqrt{-1}$ which shows that L_z contains the constant functions.

Since the basis of Lauricella functions will be used as a 'coordinate'-type map from V_n° to $\mathbb{C}^n$ while the basis we just defined in Theorem 1.2 is not that ideal, we need to modify this basis by a scalar factor as follows

$$\begin{aligned} F_k(z,\delta) &:= \int_{\delta_k} (\xi - z_0)^{-\mu_0} \cdots (\xi - z_{k-1})^{-\mu_{k-1}} (z_k - \xi)^{-\mu_k} \cdots (z_n - \xi)^{-\mu_n} d\xi \\ &= \bar{w}_k \int_{\delta_k} \psi_z \end{aligned}$$

where $w_k := \exp(\sqrt{-1}\pi(\mu_0 + \cdots + \mu_{k-1}))$. We call the multivalued map $F := (F_1, \cdots, F_n)$ from V_n° to $\mathbb{C}^n$ the *Lauricella map* and its projectivization $\mathbb{P}F$ from $\mathbb{P}(V_n^\circ)$ to $\mathbb{P}^{n-1}$ the *Schwarz map* for the weight system μ.

We notice that

$$\int_{\delta_k} \psi_z = w_k F_k(z,\delta) \quad \text{and} \quad \int_{\delta_k^-} \psi_z = \bar{w}_k F_k(z,\delta)$$

since $\psi_z|\delta_k^- = \bar{w}_k^2 \psi_z|\delta_k$. So if $|\mu| = 1$, then we have

$$\sum_{k=1}^n (w_k - \bar{w}_k) F_k(z,\delta) = \sum_{k=1}^n \left(\int_{\delta_k} \psi_z - \int_{\delta_k^-} \psi_z\right) = \int_\gamma \psi_z = 2\pi\sqrt{-1},$$

where γ is a clockwise loop enclosing $\{z_0, \cdots, z_n\}$, or equivalently,

$$\sum_{k=1}^n \mathrm{Im}(w_k) F_k(z,\delta) = \pi.$$

Corollary 1.3. *If we are not in the parabolic case, then the Lauricella map F is a local isomorphism taking values in $\mathbb{C}^n - \{0\}$. In the parabolic case, the Lauricella map F factors through a local isomorphism from $\mathbb{P}(V_n^\circ)$ to the affine hyperplane $\mathbb{A}^{n-1}$ in $\mathbb{C}^n$ defined by $\sum_{k=1}^n \mathrm{Im}(w_k) F_k = \pi$.*

In order to eliminate the multivaluedness of the Lauricella map F, we lift F to some covering space of V_n°. So it's natural to examine the fundamental group of the space in question. We take $z^\circ = (z_0^\circ, \cdots, z_n^\circ)$ as a base point for $(\mathbb{C}^{n+1})^\circ$ and use the same symbol for its image in V_n°. The projection $(\mathbb{C}^{n+1})^\circ \to V_n^\circ$ naturally induces an isomorphism on fundamental groups: $\pi_1((\mathbb{C}^{n+1})^\circ, z^\circ) \cong \pi_1(V_n^\circ, z^\circ)$. This group actually can be identified with the *pure braid group with n+1 generators* by the Brieskorn's theorem [**3**] which is denoted by PBr_{n+1}.

In fact, we can find an intermediate covering space of V_n° on which the Lauricella map becomes a single-valued map. Suppose $\gamma : [0,1] \to (\mathbb{C}^{n+1})^\circ$ is a path from $z = \gamma(0)$ to $z' = \gamma(1)$, then analytic continuation along this path gives rise to an isomorphism of vector space $A_\mu(\gamma) : L_z \to L_{z'}$. This is compatible with the composition of paths: if $\tau : [0,1] \to (\mathbb{C}^{n+1})^\circ$ is a path from $z' = \tau(0)$ to $z'' = \tau(1)$, then $A_\mu(\tau \circ \gamma) = A_\mu(\tau)A_\mu(\gamma) : L_z \to L_{z''}$. Hence if γ is a loop in $(\mathbb{C}^{n+1})^\circ$ based at z°, then the map $\gamma \mapsto A_\mu(\gamma)$ yields a representation

$$\rho : \mathrm{PBr}_{n+1} \cong \pi_1((\mathbb{C}^{n+1})^\circ, z^\circ) \to \mathrm{GL}(L_{z^\circ})$$

which is called the *monodromy representation* of the Lauricella system with weight system μ. The image of this representation is thus called the *monodromy group*, which is denoted by Γ_μ, or simply Γ.

This monodromy representation gives rise to an intermediate Γ-covering $\widehat{V_n^\circ}$ of V_n° on which all the F_k's become single-valued, usually denoted by $\widehat{F_k}$. In fact, $\widehat{V_n^\circ}$ is equal to $\ker(\rho)\backslash\widetilde{V_n^\circ}$ with $\widetilde{V_n^\circ}$ being the universal covering of V_n° and we have $\Gamma \cong \pi_1(V_n^\circ)/\ker(\rho)$. A point of $\widehat{V_n^\circ}$ can be represented as a pair (z, δ), where δ is an L-slit for z, with the understanding that (z', δ') represents the same point if and only if $z - z'$ lies on the main diagonal and $\widehat{F_k}(z, \delta) = \widehat{F_k}(z', \delta')$ for all $k = 1, \cdots, n$. From this description we see right away that the basic Lauricella functions $\widehat{F}$ on $\widehat{V_n^\circ}$ define a single-valued holomorphic map

$$\widehat{F} = (\widehat{F_1}, \cdots, \widehat{F_n}) : \widehat{V_n^\circ} \to \mathbb{C}^n - \{0\}.$$

The action of Γ on $\widehat{V_n^\circ}$ is then given as follows: if $g \in \Gamma$ is represented by the loop γ_g in $\mathbb{C}^{n+1}$ at z, then $g.[(z, \delta)] = [(z, \delta(\gamma_g^{-1}))]$, where $\delta(\gamma_g^{-1})$ means the L-slit for z after an inverse loop transformation of γ_g. We then have the following commutative diagram

$$\begin{array}{ccc} \widehat{V_n^\circ} & \xrightarrow{\widehat{F}} & \mathbb{C}^n - \{0\} \\ \downarrow & & \downarrow \\ V_n^\circ & \xrightarrow{F} & \Gamma\backslash(\mathbb{C}^n - \{0\}) \end{array}$$

with the left vertical arrow an unramified Γ-covering map. We see that F becomes a Γ-equivariant map.

In fact, there exists a Hermitian form h associated with the weight system μ which induces the corresponding geometric structure on $\mathbb{C}^n$ or on a hyperplane $\mathbb{A}^{n-1}$, depending on whether $|\mu|$ is integral. We next show how to construct such a Hermitian form h. First let $\tilde{h}$ be the Hermitian form on $\mathbb{C}^{n+1}$ defined by

$$\tilde{h}(F,G) = \sum_{1\le j<k\le n+1} \mathrm{Im}(w_j\bar{w}_k)F_k\bar{G}_j.$$

The $\tilde{h}$-orthogonal complement in $\mathbb{C}^{n+1}$ of the last basis vector e_{n+1} is the hyperplane $\mathbb{A}^n \subset \mathbb{C}^{n+1}$ defined by $\sum_{k=1}^{n+1}\mathrm{Im}(w_k)F_k = 0$. Then we have the composite map

$$pr : \mathbb{A}^n \hookrightarrow \mathbb{C}^{n+1} = \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$$

where the second map is a projection.

When $|\mu| \notin \mathbb{Z}$, we have $\mathrm{Im}(w_{n+1}) \neq 0$ and pr thus is an isomorphism. We let h be the restriction of $\tilde{h}$ to $\mathbb{A}^n$ and then transferred to $\mathbb{C}^n$ via this isomorphism. If $|\mu| \in \mathbb{Z}$, then $\mathrm{Im}(w_{n+1}) = 0$ and hence $\ker(pr) = \mathbb{C}e_{n+1}$ and $\mathrm{Im}(pr) = \mathbb{A}^{n-1} \subset \mathbb{C}^n$. Since e_{n+1} is $\tilde{h}$-isotropic, we thus obtain an induced Hermitian form on $\mathbb{A}^{n-1}$. Therefore, we construct a Γ-invariant Hermitian form h on the corresponding space. When $0 < |\mu| \le 1$, the form h is positive definite. While when $1 < |\mu| < 2$, the form h is of hyperbolic signature and we have $h(F(z,\delta), F(z,\delta)) = N(z)$, where

$$N(z) = -\frac{\sqrt{-1}}{2}\int_{\mathbb{C}} \psi_z \wedge \bar{\psi}_z = -\int_{\mathbb{C}} |z_0-\xi|^{-2\mu_0}\cdots|z_n-\xi|^{-2\mu_n}d(\text{area}).$$

This Hermitian form induces a geometric structure on the corresponding space according to the associated weight system μ. Let's have a look how the geometric structure induced by a Hermitian form. Suppose W is a finite dimensional complex vector space, then the tangent space $T_p\mathbb{P}(W)$ of its projective space $\mathbb{P}(W)$ at p can be identified with $\mathrm{Hom}(L_p, W/L_p)$; here L_p stands for a one-dimensional subspace of W which drops to $p \in \mathbb{P}(W)$. If a Hermitian form h is given on W which is nonzero on L_p, then it determines a Hermitian form h_p on $\mathbb{P}(W)$. We first identify W/L_p with $L_p^\perp$ with respect to h and think of a tangent vector as a linear map $\varphi : L_p \to L_p^\perp \cong W/L_p$. Once we choose a generator $w \in L_p$, define $h_p(\varphi,\varphi') := |h(w,w)|^{-1}h(\varphi(w),\varphi'(w))$. This is clearly independent of the generator w, so h_p is a Hermitian form on $\mathbb{P}(W)$ we desire.

If h is positive definite, then so is h_p for every $p \in \mathbb{P}(W)$. By this $\mathbb{P}(W)$ acquires a so-called *Fubini-Study* metric. When h is of hyperbolic signature, we have the set of $p \in \mathbb{P}(W)$ for which h is negative on L_p, then h_p is positive definite on this set as well. We denote this set by $\mathbb{B}(W)$ because we can write the Hermitian form h on W in the form $h(w,w) = -|w_0|^2 + |w_1|^2 + \cdots + |w_n|^2$, and then $\mathbb{B}(W)$ is defined as: $h(w,w) = -|w_0|^2 + |w_1|^2 + \cdots + |w_n|^2 < 0$, or equivalently, $|w_1/w_0|^2 + \cdots + |w_n/w_0|^2 < 1$. This is simply the open unit ball in complex m-space. Thus we call $\mathbb{B}(W)$ a *complex-hyperbolic space* and the corresponding metric *complex-hyperbolic metric.*

We conclude the above discussion as the following theorem.

Theorem 1.4. *The weight system μ endows $\mathbb{P}(V_n^\circ)$ with a natural* Kähler *metric which is Γ-invariant, and locally isometric to a Fubini-Study metric, a flat metric and a complex-hyperbolic metric when $|\mu| < 1$, $|\mu| = 1$ and $|\mu| > 1$ respectively. We call these 3 cases* elliptic, parabolic *and* hyperbolic *respectively.*

Then it is natural to raise such a question: what will happen when points coalesce (i.e., some z_k's of z come together)? Or let's phrase it in a different way: whether those geometric structures can be extended across the complementary set of V_n° in V_n? In order to deal with this issue, we need to invoke the finiteness condition of the monodromy group which is essentially controlled by the weight system μ. In fact, the Γ-covering $\widehat{V_n^\circ} \to V_n^\circ$ can be extended as a ramified Γ-covering over a bigger open subset V_n^f of V_n when the finiteness condition of the monodromy group are endowed; here V_n^f stands for the subset of V_n over which the covering map has a finite ramification. The standard tool used here is often referred to as *normalization.*

Let $\mathcal{Q}_\mu^\circ$ denote the $\mathrm{SL}_2(\mathbb{C})$-orbit space of the subset of $(\mathbb{P}^1)^{n+2}$ parameterizing pairwise distinct $(n+2)$-tuples in $\mathbb{P}^1$. The point $z = (z_0, \cdots, z_{n+1})$ is called μ-stable if the $\mathbb{R}$-divisor $\mathrm{Div}(z) := \sum_{k=0}^{n+1} \mu_k(z_k)$ has no point of weight ≥ 1. We also use S_μ to denote the group of permutations of $\{0, \cdots, n+1\}$ preserving the weights, then we should view the Lauricella map F as being multivalued on $S_\mu \backslash V_n^\circ$, rather than on V_n°. After a very careful analysis around those strata of $V_n - V_n^\circ$ which parameterize some components of z being the same, we arrive at the main result of this theory, due to Deligne-Mostow [**11**] and Mostow [**27**].

Theorem 1.5 (Deligne-Mostow)**.** *Suppose that μ satisfies the* half integrality condition*, i.e., whenever for $0 \leq k < l \leq n+1$ we have $\mu_k + \mu_l < 1$, then $1 - \mu_k - \mu_l$ is the reciprocal of an integer or the reciprocal of half an integer in case $\mu_k = \mu_l$.*

ell: *If $|\mu| < 1$, then Γ is a finite complex reflection group in* $\mathrm{GL}(n,\mathbb{C})$ *and the Lauricella map $\widehat{F} : \widehat{S_\mu \backslash V_n} \to \mathbb{C}^n - \{0\}$ is a Γ-equivariant isomorphism and thus descends to an isomorphism $F : S_\mu \backslash V_n \to \Gamma \backslash (\mathbb{C}^n - \{0\})$.*

par: *If* $|\mu| = 1$, *then* Γ *acts as a complex* Bieberbach group *in* $\mathbb{A}^{n-1}$ *and the Schwarz map* $\widehat{\mathbb{P}F} : \mathbb{P}(\widehat{S_\mu \backslash V_n}) \to \mathbb{A}^{n-1}$ *is a* Γ*-equivariant isomorphism and thus descends to an isomorphism* $\mathbb{P}F : \mathbb{P}(S_\mu \backslash V_n) \to \Gamma \backslash \mathbb{A}^{n-1}$.

hyp: *If* $1 < |\mu| < 2$, *then the* Γ*-covering* $\widehat{S_\mu \backslash \mathcal{Q}_\mu^\circ} \to S_\mu \backslash \mathcal{Q}_\mu^\circ$ *extends to a ramified covering* $\widehat{S_\mu \backslash \mathcal{Q}_\mu^{\mathrm{st}}} \to S_\mu \backslash \mathcal{Q}_\mu^{\mathrm{st}}$ *and* $\widehat{\mathbb{P}F}$ *extends to a* Γ*-equivariant isomorphism* $\widehat{S_\mu \backslash \mathcal{Q}_\mu^{\mathrm{st}}} \to \mathbb{B}^{n-1}$. *Moreover* Γ *acts discretely in this complex ball and with finite covolume.*

The reader can consult [**23**] for a detailed discussion on this result.

Deligne and Mostow also gave a modular interpretation of their ball quotients. Of course we are doing this discussion under the hyperbolic case with discrete monodromy: $\mu_k \in (0,1)$ and rational for $k = 0, \cdots, n+1$ such that $1 < |\mu| < 2$. From this point of view, the Schwarz map can be interpreted as a period map. The idea is to lift $\mathbb{P}^1$ to a cyclic cover on which the Lauricella integrand becomes a regular differential. In order to see this, we first write $\mu_k = m_k/m$ as quotient of positive integers such that $\gcd(m, m_0, \cdots, m_{n+1}) = 1$ and let d_k be the denominator of the reduced fraction μ_k. Consider the curve given by

$$\zeta^m = \prod_{k=0}^{n} (z_k - \xi)^{m_k}$$

in affine coordinates. This is a cyclic cover $C \to \mathbb{P}^1$ of order m ramified over z_k of order d_k, with a regular holomorphic differential $\psi = d\xi/\zeta$ on it. The periods

$$\int_{z_k}^{z_{k+1}} \psi$$

are also a basis of Lauricella functions. Therefore, the Schwarz map $\widehat{\mathbb{P}F} : \widehat{\mathcal{Q}_\mu^{st}} \to \mathbb{B}^{n-1}$ can now be understood as a period map associated to the curve C. The similar example in the toric case will be investigated in more detail in Section 3.6.

1.2. Geometric structures on projective arrangement complements

Inspired by the idea that viewing the configuration space of $n+1$ pairwise distinct points in $\mathbb{C}$ as a complement of a (linear) hyperplane arrangement in $\mathbb{C}^{n+1}$ and the observation that hypergeometric functions actually give rise to a local system on $(\mathbb{C}^{n+1})^\circ$, Couwenberg, Heckman and Looijenga [**6**] extended the Deligne-Mostow theory to a more general setting, which is of a stronger geometric nature.

Let us start with a finite dimensional complex vector space V endowed with an inner product, a finite collection $\mathcal{H}$ of linear hyperplanes in V. Such $(V,\mathcal{H})$ is called a *linear hyperplane arrangement.* We define the *arrangement complement* by $V^\circ := V - \cup_{H\in\mathcal{H}}H$, i.e., the complement of the union of the members of $\mathcal{H}$ in V. We shall always use the superscript $^\circ$ to denote the complement of an arrangement in an analogous situation as long as the arrangement is understood.

We denote by $\mathcal{L}(\mathcal{H})$ the collection of hyperplane intersections taken from subsets of $\mathcal{H}$; here we understand V is also included in $\mathcal{L}(\mathcal{H})$ as the intersection over the empty subset of $\mathcal{H}$. And for $L\in\mathcal{L}(\mathcal{H})$, we denote by $\mathcal{H}_L$ the collection of $H\in\mathcal{H}$ which contains L. On the other hand, each $H\in\mathcal{H}-\mathcal{H}_L$ meets L in a hyperplane of L. We denote the collection of these hyperplanes of L by $\mathcal{H}^L$. But we notice that the natural map $\mathcal{H}-\mathcal{H}_L\to\mathcal{H}^L$ needs not be injective, so that $\mathcal{H}-\mathcal{H}_L$ and $\mathcal{H}^L$ cannot be identified in general. We say that $\mathcal{H}$ is *irreducible* if there does not exist a nontrivial decomposition of $\mathcal{H}$ such that $\mathcal{H}=\mathcal{H}_L\sqcup\mathcal{H}_{L'}$ with $L,L'\in\mathcal{L}(\mathcal{H})$ and $L\oplus L'=V$. A member $L\in\mathcal{L}(\mathcal{H})$ is also called irreducible if $\mathcal{H}_L$ is. We denote the subset of irreducible members of $\mathcal{L}(\mathcal{H})$ by $\mathcal{L}_{\mathrm{irr}}(\mathcal{H})$. For more about hyperplane arrangement, the interested reader could consult Orlik and Terao [**31**].

Suppose we are given a map κ which assigns to every $H\in\mathcal{H}$ a positive real number k_H, then we can define a connection ∇^κ on the tangent bundle of V°. For $H\in\mathcal{H}$, let $\pi_H\in\mathrm{End}(V)$ denote the orthogonal projection onto $H^\perp$, then $\rho_H:=k_H\pi_H$ is self-adjoint with respect to the inner product. So its kernel is H with trace k_H. We also let ω_H denote the unique meromorphic differential on V with divisor $-H$ and residue 1 along H. So $\omega_H:=\phi_H^{-1}d\phi_H$, where ϕ_H is a defining linear equation for H. Put the *connection form*

$$\Omega^\kappa := \sum_{H\in\mathcal{H}}\omega_H\otimes\rho_H$$

and view it as an $\mathrm{End}(V)$-valued holomorphic differential on V°. Then the desired connection is defined as follows

$$\nabla^\kappa := \nabla^0 - \Omega^\kappa$$

where ∇^0 denotes the standard (translation invariant) connection on V restricted on V°.

Now our ∇^κ is easily verified to be torsion free and there are some criteria for the flatness of this connection.

Theorem 1.6. ∇^κ *is* $\mathbb{C}^\times$*-invariant and torsion free. Moreover, the following properties are equivalent:*

(i) ∇^κ *is flat,*
(ii) $\Omega^\kappa\wedge\Omega^\kappa=0,$

(iii) *for every pair* $L, M \in \mathcal{L}(\mathcal{H})$ *with* $L \subset M$, *the endomorphisms* $\sum_{H\in\mathcal{H}_L} \rho_H$ *and* $\sum_{H\in\mathcal{H}_M} \rho_H$ *commute,*
(iv) *for every* $L \in \mathcal{L}(\mathcal{H})$ *of codimension* 2, *the sum* $\sum_{H\in\mathcal{H}_L} \rho_H$ *commutes with each of its terms.*

PROOF. See [**21**] and [**6**]. A less general form of the flatness criterion also appeared in a paper by Kohno [**19**]. We shall later also prove part of these criteria when we need them along the way where we deal with the toric case. □

If the connection ∇^κ is flat, we say that the triple $(V, \mathcal{H}, \kappa)$ is a *Dunkl system.* From [**6**] we can see that besides in the Lauricella case there also exist many other Dunkl systems of interest, such as in the complex reflection case.

It is well-known that a flat torsion free connection on a tangent bundle defines an affine structure. We denote the holonomy group of the Dunkl system by Γ. So both V° and its Γ-covering $\widehat{V^\circ}$ come with one, denoted by Aff_{V° resp. $\mathrm{Aff}_{\widehat{V^\circ}}$, that is a subsheaf of affine-linear functions of the structure sheaf. The sheaf of affine-linear functions is in fact the sheaf of holomorphic functions whose differential is flat for the connection, which corresponds to a system of second order differential equations. Conversely, an affine structure is always given by a flat torsion free connection. Then we see that the space of affine-linear functions on $\widehat{V^\circ}$ is given by $\mathrm{Aff}(\widehat{V^\circ}) := H^0(\widehat{V^\circ}, \mathrm{Aff}_{\widehat{V^\circ}})$. We also denote by A the set of linear forms $\mathrm{Aff}(\widehat{V^\circ}) \to \mathbb{C}$ which are the identity on $\mathbb{C}$. In fact, this is an affine hyperplane in $\mathrm{Aff}(\widehat{V^\circ})^*$. Then the evaluation map $ev : \widehat{V^\circ} \to A$ which assigns to $\hat{z}$ the linear form $ev_{\hat{z}} : \hat{f} \in \mathrm{Aff}(\widehat{V^\circ}) \mapsto \hat{f}(\hat{z}) \in \mathbb{C}$ is called the *developing map* of the affine structure.

Then for each given κ, we can deform the connection and the Hermitian form at the same time so that we can induce different geometric structures on $\mathbb{P}(V^\circ)$. First we can deform the standard connection ∇^0 in a real analytic way to a one-parameter family of flat torsion free connections, denoted by ∇^s, $s \geq 0$. On the other hand, the inner product gives rise to a translation invariant metric on V. Its restriction h^0 to V° is flat for ∇^0. Then we can also deform h^0 to a family of Hermitian forms such that each h^s is flat for ∇^s. Since the scalar multiplication in V acts locally like homothety, we have that $\mathbb{P}(V^\circ)$ inherits a Hermitian metric g^s from V°. We only allow s vary in an interval for which g^s stays positive definite. Meanwhile this still makes it possible for h^s to become degenerate or of hyperbolic signature as long as for every $p \in V^\circ$, the restriction of h^s to a hyperplane which is supplementary and perpendicular to $T_p(\mathbb{C}p)$ is positive definite. Then the geometric structures could be classified into following 3 cases: if h^s is positive on $T_p(\mathbb{C}p)$, then we are given a Fubini-Study metric g^s on $\mathbb{P}(V^\circ)$, we refer to this situation as the *elliptic* case; if $T_p(\mathbb{C}p)$ is the kernel of h^s, then g^s is a flat metric, we refer to this situation as the *parabolic* case; if h^s is negative on $T_p(\mathbb{C}p)$, then g^s is locally the metric of a complex ball, we refer to this situation as the *hyperbolic* case.

We conclude this discussion as the following theorem.

Theorem 1.7. *Suppose we are given a κ such that ∇^κ is flat and a corresponding flat Hermitian form h^κ is of parabolic type. For every $s \geq 0$, we are given a nonzero Hermitian form h^s on V° which is flat for $\nabla^{s\kappa}$. Assume it is real-analytic in s and is equal to the given positive definite form for $s = 0$. Then we have:*

(i) for $s < 1$, $h^s > 0$,
(ii) $h^1 \geq 0$,
(iii) there exists a $m_{\text{hyp}} \in (1, \infty]$ such that for $s \in (1, m_{\text{hyp}})$, h^s is of hyperbolic type.

We call m_{hyp} the hyperbolic exponent *of the family.*

See [**6**] for a detailed discussion about this result. Then a given pair $(\nabla^{s\kappa}, h^s)$ endows V° with a geometric structure. If we want to extend this structure across the arrangement, we have to do a very careful analysis near those components of the arrangement. Fortunately, we do have some nice hereditary properties of the Dunkl connection. For $L \in \mathcal{L}(\mathcal{H})$ irreducible, since $\sum_{H\in\mathcal{H}_L} k_H\pi_H$ commutes with each of its terms, by Schur's lemma, we must have that $\sum_{H\in\mathcal{H}_L} k_H\pi_H = k_L\pi_L$ for some k_L, where π_L is the orthogonal projection with kernel L. So by a trace computation, we have $k_L = \operatorname{codim}(L)^{-1}\sum_{H\in\mathcal{H}_L} k_H$. Hence for the origin, which is viewed as an intersection of all the hyperplanes (suppose we have enough hyperplanes in general positions), we have $k_0 = \dim(V)^{-1}\sum_{H\in\mathcal{H}} k_H$. In fact, these numbers k_L have an inclusion-reverse property, i.e., $k_M < k_L$ if $L \subsetneq M$. If we denote by D the exceptional divisor of the blow-up of L in V, then the affine structure on V° is of *infinitesimally simple type* along D° (c.f. Definition 3.4) with logarithmic exponent $k_L - 1$.

Then for every $L \in \mathcal{L}(\mathcal{H})$, put

$$\Omega_L := \sum_{H\in\mathcal{H}_L} \omega_H \otimes \rho_H.$$

This defines a Dunkl connection ∇_L on $(V/L)^\circ$.

Let i_L denote the inclusion: $L \subset V$. If I is given by $L \cap H$ for some $H \in \mathcal{H} - \mathcal{H}_L$, then $\omega_I^L := i_L^*(\omega_H)$ is the logarithmic differential on L defined by I. Also notice that for each $I \in \mathcal{H}^L$, there exists a unique irreducible intersection $I(L) \in \mathcal{L}(\mathcal{H})$ such that $L \cap I(L) = I$. Then we can also define a Dunkl connection ∇^L on L° as follows

$$\Omega^L := \sum_{I\in\mathcal{H}^L} \omega_I^L \otimes k_{I(L)}\pi_I^L$$

where π_I^L denotes the restriction of π_I to L.

We thus call the Dunkl connection ∇_L resp. ∇^L defined on $(V/L)^\circ$ resp. L° the *L-transversal* resp. *L-longitudinal* Dunkl connection.

Now in order to extend the geometric structure across the arrangement "nicely", we have to impose the so-called *Schwarz condition* on the Dunkl system. But before we proceed to that, let us have a look at a simple example first so that we can have some feeling why the Schwarz condition are introduced in the way as later. It is a one-dimensional example. Let V be $\mathbb{C}$, $\mathcal{H}$ consists of the origin and $\Omega = k\frac{dz}{z}$. Assume we have finite holonomy, which means we can write $1-k = p/q$ with p, q relatively prime integers and $q > 0$. Then the holonomy cover can be extended to a q-fold cover with ramification over the origin $\hat{V} \to V$ defined by $\hat{z}^q = z$. On the other hand, the developing map $\hat{V} - \{0\} \to \mathbb{C}$ is given by $w = \hat{z}^p$ and hence extends across the origin only if $p > 0$, i.e., $k < 1$. But we could note that the connection is invariant under the pth roots of unity ξ_p which means the ξ_p-orbit space of V is covered by the ξ_p-orbit space of $\hat{V}$ and the developing map factors through the latter as an isomorphism onto $\mathbb{C}$.

This example suggests the definition for Schwarz condition. Assume $1 - k_L \neq 0$ can be written as a fraction p_L/q_L with p_L, q_L relatively prime integers and $q_L > 0$. We call G_L the *Schwarz rotation group* of L, which is the subgroup of the unitary group $U(V)$ consisting of elements fixing L pointwise and acting as scalar multiplication in $L^\perp$ by a $|p_L|$th root of unity. Then we say that $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ satisfies the Schwarz condition if the Dunkl system is invariant under G_L. We say that the Dunkl system satisfies the Schwarz condition if every $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ satisfies the Schwarz condition. We denote by G the *Schwarz symmetry group* which is the subgroup of $U(V)$ generated by the Schwarz rotation group G_L of the $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ satisfying the Schwarz condition.

Suppose now the Dunkl system satisfies the Schwarz condition. As illustrated by the previous one-dimensional example, for $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$, it's easy to extend the developing map across L° when $1 - k_L > 0$. But when $1 - k_L \leq 0$, the situation becomes quite different because if we approach L° from V° along a curve, the image of a lift in $\widehat{V^\circ}$ under the developing map tends to infinity with limit a point of $\mathbb{P}(A)$. In fact, these limit points lie in a well-defined Γ-orbit of linear subspaces of $\mathbb{P}(A)$ of codimension $\dim(L)$, which is called a *special subspace* in $\mathbb{P}(A)$. So we say that V° has geometric structures of *elliptic, parabolic, hyperbolic* type according to whether $k_0 < 1$, $= 1$ or > 1. That is because $k_0 < 1$ (resp. $k_0 = 1$) ensures $k_L < 1$ for all $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ (resp. $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H}) - \{0\}$) due to the monotonicity of k_L's.

While the most interesting case is the one of hyperbolic type, we need to treat L° with $k_L \geq 1$ very carefully. Now we assume the Dunkl system with the flat Hermitian form h is of the hyperbolic type. We notice that the restriction of h to the fibers of the natural retraction $r : V_{L^\circ} \to L^\circ$ is positive, semipositive and hyperbolic according to whether $1 - k_L > 0$, $= 0$ or < 0. From this we see that when $k_L < 1$ L° still keeps its hyperbolic type while other cases not. So for each $L \in \mathcal{L}_{\text{irr}}(\mathcal{H})$ with $k_L \geq 1$ we need to blow it up and contract it in its own direction so that each of them has a hyperbolic structure in the end. Of course, we need to blow up those L's in an appropriate order. And we should point out that for those L's with $k_L = 1$, we need to blow up each of them in a *real-oriented manner.* The process of blowing up and followed by blowing down is a very technical tool, interested reader could consult [**22**] and [**6**] for a detailed and complete discussion. After we finish these operations, we can extend the corresponding structure across the arrangement. We conclude the above discussion by the following theorem.

Theorem 1.8 (Couwenberg-Heckman-Looijenga). *Suppose the Dunkl system satisfies the Schwarz condition and there is a flat Hermitian form h on the tangent bundle of V°.*

- **ell:** *if $0 < k_0 < 1$, then h is positive definite. The developing map $\widehat{ev_G} : G\backslash\hat{V} \to A$ is a Γ-equivariant isomorphism and thus descends to an isomorphism of orbit space of reflection groups $ev_G : G\backslash V \to \Gamma\backslash A$. Moreover, $\mathbb{P}(G\backslash V)$ acquires a new structure as a complete elliptic orbifold via ev_G.*
- **par:** *if $k_0 = 1$, then h is positive semidefinite with kernel K on the translation space of A. The developing map $\widehat{ev_G} : G\backslash \widehat{V - \{0\}} \to A$ is a Γ-equivariant isomorphism and thus descends to an isomorphism of orbit space of complex crystallographic groups $ev_G : G\backslash V \to \Gamma\backslash(K\backslash A)$. Moreover, $\mathbb{P}(G\backslash V)$ acquires the structure of a complete parabolic orbifold via ev_G.*
- **hyp:** *if $1 < k_0 < m_{\text{hyp}}$ for some $m_{\text{hyp}} > 1$, then h is of hyperbolic type so that h defines a complex ball $\mathbb{B}$ in the projective space $\mathbb{P}(A)$. If $A^\diamond$ denotes the complement of the union of the special hyperplanes in A, then the developing map $\widehat{ev_G} : G\backslash\widehat{V_{k<1}} \to A^\diamond$ is a Γ-equivariant isomorphism and thus descends to an isomorphism $ev_G : G\backslash V_{k<1} \to \Gamma\backslash A^\diamond$. Here $V_{k<1}$ denotes the union of the strata L° with $k_L < 1$ which is an open subset of V ($V^\circ \subset V_{k<1}$ since $k_V = 0$). Moreover, $\mathbb{P}(G\backslash V_{k<1})$ acquires the structure of a hyperbolic orbifold whose metric completion is $\Gamma\backslash\mathbb{B}$.*

Therefore, we obtain new examples of groups operating discretely and with finite covolume on a complex ball through the hyperbolic case.

Now let us see how this theory includes the Deligne-Mostow theory as a special case. We start with a weight system μ which gives rises to a Lauricella differential. Let $V := V_n = \mathbb{C}^{n+1}/\text{main diagonal}$. Denote the standard basis of $\mathbb{C}^{n+1}$ by $\varepsilon_0, \cdots, \varepsilon_n$. Let $\mathcal{H}$ be the collection of diagonal hyperplanes $H_{ij} \subset V$ defined by $z_i = z_j$ and $\omega_{ij} := (z_i - z_j)^{-1} d(z_i - z_j)$ the associated logarithmic form. The inner product on V comes from the inner product on $\mathbb{C}^{n+1}$ given by $(\varepsilon_i, \varepsilon_j) = \mu_i \delta_{ij}$. In fact, we could identify V with the orthogonal complement of the main diagonal which is the hyperplane defined by $\sum_i \mu_i z_i = 0$. The line orthogonal to the hyperplane H_{ij} is spanned by the vector $\mu_j \varepsilon_i - \mu_i \varepsilon_j$. Sometimes we replace the original basis by a new basis $\{\varepsilon_i' := \mu_i^{-1} \varepsilon_i\}$ for the reason that the hyperplane H_{ij} then is orthogonal to $\varepsilon_i' - \varepsilon_j'$; notice that the inner product changed $(\varepsilon_i', \varepsilon_j') = \mu_i^{-1} \delta_{ij}$. Now the endomorphism $\tilde{\rho}_{ij} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}; z \mapsto (z_i - z_j)(\mu_j \varepsilon_i - \mu_i \varepsilon_j)$ is selfadjoint with kernel H_{ij}. And we also have $\tilde{\rho}_{ij}(\mu_j \varepsilon_i - \mu_i \varepsilon_j) = (\mu_i + \mu_j)(\mu_j \varepsilon_i - \mu_i \varepsilon_j)$ so that $k_{H_{ij}} = \mu_i + \mu_j$. In particular, $\tilde{\rho}_{ij}$ induces an endomorphism ρ_{ij} in V. We can verify that the connection defined by

$$\nabla := \nabla^0 - \sum_{i<j} \omega_{ij} \otimes \rho_{ij}$$

is flat so that it becomes a Dunkl system. In fact, the space of affine-linear functions at $z \in V^\circ$ is precisely the space of solutions of the system of differential equations we have seen in part (iii) of Proposition 1.1. Therefore, the Schwarz map can now be understood as a projectivized developing map for the new projective structure on $\mathbb{P}(V^\circ)$. We can also compute that $k_0 = |\mu|$; and if an irreducible member $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ is given by a subset $I \subset \{0, \cdots, n\}$ with at least two elements, i.e., $L = L(I)$ is the locus where all $z_j, j \in I$ coincide, then $k_L = \sum_{j \in I} \mu_j$. A further discussion, like how the Schwarz condition appears in the case, can be found in Example 3.26 which deals with its toric analogue.

We shall in this thesis use the point of view introduced in this section to reinvestigate the special hypergeometric system associated with a root system.

1.3. Torus embeddings

Since we shall treat the arrangement complement for the toric case in this thesis, we feel it necessary to invest some pages here to give a brief introduction to the theory of torus embeddings. In fact, the name “torus embeddings” is somehow an outdated one, people nowadays more and more use the name “toric varieties” instead. Simply speaking, a toric variety is a normal variety X that contains a torus T as a dense open subset, together with an action $T \times X \to X$ of T on X that extends the natural action of T on itself. From this viewpoint, we can see that the torus action is as important as compactifying a torus in this theory. That is why the name “toric varieties” is more popular today, which gives a more complete description on this theory. However, we shall still use the name “torus embeddings” here because we emphasize the compactification problem rather than the torus action in this thesis. An introduction to this theory can be found in Oda [**28**] and Fulton [**12**].

Let be given a free $\mathbb{Z}$-module N of rank n, i.e., $N \cong \mathbb{Z}^n$. Let M be its dual $\mathbb{Z}$-module defined by $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Then we have a canonical bilinear pairing

$$(\cdot, \cdot) : M \times N \to \mathbb{Z}.$$

We can extend this to the real situation by scalar extension to the real number field $\mathbb{R}$. Thus we have n-dimensional real vector spaces $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ with a canonical $\mathbb{R}$-bilinear pairing

$$(\cdot, \cdot) : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}.$$

A subset σ of $N_{\mathbb{R}}$ is called a *strongly convex rational polyhedral cone* (with apex at the origin O) if it is a cone generated by a finite number of vectors in N such that $\sigma \cap (-\sigma) = \{O\}$. In other words, if we denote the finite number of vectors in N by $v_1, \cdots, v_r$, we can define σ as

$$\sigma = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_r$$

containing no line through the origin.

The dual cone in $M_{\mathbb{R}}$ of σ is the set of vectors of $M_{\mathbb{R}}$ that are ≥ 0 on σ

$$\sigma^{\vee} := \{u \in M_{\mathbb{R}} \mid (u, v) \geq 0 \text{ for all } v \in \sigma\}.$$

So if $v_0 \notin \sigma$, then there must be some $u_0 \in \sigma^{\vee}$ such that $(u_0, v_0) < 0$. Based on this fundamental fact about the convex set, we have the following elementary properties of σ. Note that a subset τ of σ is called a *face* and is denoted $\tau \prec \sigma$, if

$$\tau = \sigma \cap \{u_0\}^{\perp} := \{v \in \sigma \mid (u_0, v) = 0\}$$

for a $u_0 \in \sigma^{\vee}$. A face of codimension one is called a *facet.*

Lemma 1.9. *(i)* $(\sigma^{\vee})^{\vee} = \sigma$;
(ii) Any face is also a convex polyhedral cone;
(iii) Any intersection of faces is also a face;
(iv) Any face of a face is a face;
(v) Any proper face is contained in some facet.

In fact, this lemma holds as long as σ is a convex polyhedral cone.

Proposition 1.10 (Gordon's Lemma)**.** *If σ is a rational convex polyhedral cone, then $S_\sigma = \sigma^{\vee} \cap M$ is a finitely generated semigroup.*

Any additive finitely generated semigroup S gives rise to a commutative $\mathbb{C}$-algebra $\mathbb{C}[S]$. As a complex vector space it has a basis e^u, as u varies over S, with multiplication corresponding to the addition in S:

$$e^u \cdot e^{u'} = e^{u+u'}.$$

It's clear that the unit 1 is e^0 and generators $\{u_i\}$ for the semigroup S give rise to generators $\{e^{u_i}\}$ for the $\mathbb{C}$-algebra $\mathbb{C}[S]$. For $A = \mathbb{C}[S]$, the closed points of

$\mathrm{Spec}(A)$ actually correspond to homomorphisms of semigroups $S \to \mathbb{C}$ where $\mathbb{C} = \mathbb{C}^\times \cup \{0\}$ is regarded as an abelian semigroup with multiplication:

$$\mathrm{Specm}(\mathbb{C}[S]) = \mathrm{Hom}_{\mathrm{sg}}(S, \mathbb{C}).$$

Since e^u is a *character* which gives rise to a homomorphism $e^u : \mathrm{Specm}(\mathbb{C}[S]) \to \mathbb{C}^\times$, the value of e^u at the corresponding point of $\mathrm{Specm}(\mathbb{C}[S])$ which is a homomorphism x from S to $\mathbb{C}$ can be defined by: $e^u(x) = x(u)$.

When $S = S_\sigma = \sigma^\vee \cap M$ arises from a strongly convex rational polyhedral cone σ, put $A_\sigma = \mathbb{C}[S_\sigma]$, and define

$$U_\sigma := \mathrm{Spec}(\mathbb{C}[S_\sigma]) = \mathrm{Spec}(A_\sigma)$$

to be the corresponding *affine toric variety*. In fact, from the above point of view, the set of the closed points of U_σ (denoted by U_σ also) can be described in a slightly different way. If S_σ can be written as $S_\sigma = \mathbb{Z}_{\geq 0}u_1 + \cdots + \mathbb{Z}_{\geq 0}u_r$, let

$$U_\sigma := \{x : S_\sigma \to \mathbb{C} \mid x(O) = 1, x(u + u') = x(u)x(u'),\ \forall u, u' \in S_\sigma\}$$

and let $e^u(x) := x(u)$ for $u \in S_\sigma$ and $x \in U_\sigma$. Then the map

$$(e^{u_1}, \cdots, e^{u_r}) : U_\sigma \to \mathbb{C}^r = \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{\text{r copies}}$$

defines a bijection from U_σ onto its image in $\mathbb{C}^r$.

If we use the language of algebraic geometry, the structure of an algebraic variety on U_σ can be described very concisely. We notice that $M = S_{\{O\}}$. Let $\mathbb{C}[M] := \bigoplus_{u \in M} \mathbb{C}e^u$ be the group algebra of M over $\mathbb{C}$, where e^u are defined as above. The ring multiplication is defined by $e^u \cdot e^{u'} := e^{u+u'}$ for $u, u' \in M$. If $v_1, \cdots, v_n$ is a basis of N, and $u_1, \cdots, u_n$ is the dual basis of M, put $X_i = e^{u_i} \in \mathbb{C}[M]$. Since as a semigroup M has generators $\pm u_1, \cdots, \pm u_n$, we have

$$\mathbb{C}[M] = \mathbb{C}[X_1, X_1^{-1}, \cdots, X_n, X_n^{-1}].$$

So

$$U_{\{O\}} = \mathrm{Spec}(\mathbb{C}[M]) \cong \underbrace{\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times}_{\text{n copies}} = (\mathbb{C}^\times)^n$$

is an affine *algebraic torus*. All of our semigroups S_σ are additive subsemigroups of M, so $\mathbb{C}[S_\sigma]$ is a $\mathbb{C}$-subalgebra of $\mathbb{C}[M]$ with $\{e^u \mid u \in S_\sigma\}$ as a basis. Then U_σ obviously coincides with the set of closed points of the affine scheme $\mathrm{Spec}(\mathbb{C}[S_\sigma])$, with a morphism $\mathrm{Spec}(\mathbb{C}[M]) \to \mathrm{Spec}(\mathbb{C}[S_\sigma])$.

This morphism already gives a hint on that how to glue those affine toric varieties associated to different faces together. But before we proceed to that, let's first have a look at the algebraic torus, a fundamental class of toric varieties, in more detail. The torus $T = T_N$ corresponding to N can be written intrinsically:

$$T_N = \mathrm{Spec}(\mathbb{C}[M]) = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times) = N \otimes_{\mathbb{Z}} \mathbb{C}^\times.$$

Each $u \in M$ gives rise to a character e^u, which is a homomorphism $e^u : T_N \to \mathbb{C}^\times$ defined by $e^u(p) := p(u)$ for $p \in T_N$. We also have the exponential law $e^{u+u'} = e^u e^{u'}$ for $u, u' \in M$. In particular, We have $e^O = 1$ and can identify M with the character group of T_N.

On the other hand, each $v \in N$ gives rise to a *one-parameter subgroup* $\gamma_v : \mathbb{C}^\times \to T_N$, which is a homomorphism defined by $\gamma_v(t)(u) := t^{(u,v)}$ for $t \in \mathbb{C}^\times$ and $u \in M$. We also have $\gamma_{v+v'} = \gamma_v \cdot \gamma_{v'}$ for $v, v' \in N$, so that we can identify N with the group of one-parameter subgroups of T_N.

We can also explain the above expression in an explicit way. Choose a basis $\{v_1, \cdots, v_n\}$ of N and let $\{u_1, \cdots, u_n\}$ be the dual basis of M. If we denote the character e^{u_i} by x_i, then we have an isomorphism

$$T_N \xrightarrow{\sim} (\mathbb{C}^\times)^n$$

which sends p to $(x_1(p), \cdots, x_n(p))$. Thus $(x_1, \cdots, x_n)$ can be regarded as a coordinate system for T_N. For $u = \sum_{1 \le i \le n} a_i u_i$, we have $e^u = x_1^{a_1} \cdots x_n^{a_n}$. On the other hand, $v = \sum_{1 \le i \le n} b_i v_i$ gives rise to a homomorphism

$$\gamma_v : \mathbb{C}^\times \to T_N$$

which sends t to $(t^{b_1}, \cdots, t^{b_n}) \in (\mathbb{C}^\times)^n \cong T_N$.

Note that an algebraic torus is a commutative algebraic group.

First let us see some examples for which how a strongly convex rational polyhedral cone gives rise to an affine toric variety.

Example 1.11. Suppose $n = 2$. Choose a basis $\{v_1, v_2\}$ of N and let $\{u_1, u_2\}$ be the dual basis of M.

(i) If $\sigma = \mathbb{R}_{\ge 0} v_1 + \mathbb{R}_{\ge 0} v_2$, then $\sigma^\vee = \mathbb{R}_{\ge 0} u_1 + \mathbb{R}_{\ge 0} u_2$. Hence $S_\sigma = \mathbb{Z}_{\ge 0} u_1 + \mathbb{Z}_{\ge 0} u_2$ and we have an isomorphism $(e^{u_1}, e^{u_2}) : U_\sigma \xrightarrow{\sim} \mathbb{C}^2$.

(ii) If $\sigma = \mathbb{R}_{\ge 0} v_1$, then $\sigma^\vee = \mathbb{R}_{\ge 0} u_1 + \mathbb{R} u_2$. Thus $S_\sigma = \mathbb{Z}_{\ge 0} u_1 + \mathbb{Z}_{\ge 0} u_2 + \mathbb{Z}_{\ge 0}(-u_2)$ and the embedding $(e^{u_1}, e^{u_2}, e^{-u_2}) : U_\sigma \to \mathbb{C}^3$ gives rise to an isomorphism $U_\sigma = \{(x_1, x_2, x_2) \in \mathbb{C}^3 \mid x_2 x_3 = 1\} \xrightarrow{\sim} \{(x_1, x_2) \in \mathbb{C}^2 \mid x_2 \ne 0\} = \mathbb{C} \times \mathbb{C}^\times$.

(iii) If $\sigma = \mathbb{R}_{\ge 0} v_1 + \mathbb{R}_{\ge 0}(v_1 + 2v_2)$, then $\sigma^\vee = \mathbb{R}_{\ge 0}(2u_1 - u_2) + \mathbb{R}_{\ge 0} u_2$. Thus $S_\sigma = \mathbb{Z}_{\ge 0} u_1 + \mathbb{Z}_{\ge 0} u_2 + \mathbb{Z}_{\ge 0}(2u_1 - u_2)$. And from the embedding $(e^{u_1}, e^{u_2}, e^{2u_1 - u_2}) : U_\sigma \to \mathbb{C}^3$ we can see that $U_\sigma = \{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_1^2 = x_2 x_3\}$ which has an isolated singularity at the origin $O = \{0, 0, 0\}$.

If τ is a face of σ, then there exists a $u_0 \in \sigma^\vee \cap M$ such that $\tau = \sigma \cap \{u_0\}^\perp = \{v \in \sigma \mid (u_0, v) = 0\}$. Thus τ is also a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$ so that S_τ can be written as $S_\tau = S_\sigma + \mathbb{Z}_{\ge 0}(-u_0)$. Hence the map $\text{Spec}(\mathbb{C}[S_\tau]) = U_\tau \to U_\sigma = \text{Spec}(\mathbb{C}[S_\sigma])$ embeds U_τ as a principle open subset of U_σ. So it is natural to introduce a notion which is a collection of those strongly convex rational polyhedral cones in order to glue those pieces of affine toric varieties together to a toric variety.

Usually a nonempty collection of strongly convex rational polyhedral cones σ in $N_\mathbb{R}$ is called a *fan* $\triangle$ if it satisfies the following conditions:

(i) Every face of a cone in $\triangle$ is still a cone in $\triangle$;
(ii) The intersection of two cones in $\triangle$ is a face of each cone.

The union $|\triangle| := \cup_{\sigma\in\triangle}\sigma$ is called the *support* of $\triangle$.

Now we are prepared to construct a toric variety from simplicial geometry.

Associate to a fan $\triangle$ the toric variety $X(\triangle)$ can be constructed as follows: first taking the disjoint union of the affine toric varieties U_σ, one for each cone $\sigma \in \triangle$ and then gluing in a way which has already been mentioned: for cones σ and τ, since the intersection $\sigma \cap \tau$ is a face of each of σ and τ and is thus still a cone in $\triangle$, so that $U_{\sigma\cap\tau}$ is identified as a principle open subvariety of U_σ and of U_τ, then we can glue U_σ and U_τ via the identification on $U_{\sigma\cap\tau}$. These identifications are compatible because the correspondence from cones to affine varieties is order-preserving. And the constructed variety is separated because the diagonal map $U_{\sigma\cap\tau} \to U_\sigma \times U_\tau$ is closed.

Example 1.12. Suppose $n = 1$. Then $N = \mathbb{Z}$.

(i) If $\sigma := \mathbb{R}_{\geq 0} \subset N_\mathbb{R}$, then $\triangle := \{\sigma, \{O\}\}$ is a fan. We obtain $X(\triangle)$ by embedding $U_{\{O\}} = \mathbb{C}^\times$ into $U_\sigma \cong \mathbb{C}$.

(ii) If $\sigma := \mathbb{R}_{\geq 0}$, then $\triangle := \{\sigma, -\sigma, \{O\}\}$ is a fan. We obtain $X(\triangle)$ by gluing $U_\sigma \cong \mathbb{C}$ and $U_{-\sigma} \cong \mathbb{C}$ along their common open subset $U_{\{O\}} = \mathbb{C}^\times$. It's just the complex projective line $\mathbb{P}^1(\mathbb{C})$.

Example 1.13. Suppose $n = 2$. Choose a basis $\{v_1, v_2\}$ of N and let $\{u_1, u_2\}$ be the dual basis of M.

(i) Let $\sigma_1 := \mathbb{R}_{\geq 0}v_1$ and $\sigma_2 := \mathbb{R}_{\geq 0}v_2$, then $\triangle = \{\sigma_1, \sigma_2, \{O\}\}$ is a fan. We obtain $X(\triangle) = \mathbb{C}^2\backslash\{(0,0)\}$ by gluing $U_{\sigma_1} = \mathbb{C} \times \mathbb{C}^\times$ and $U_{\sigma_2} = \mathbb{C}^\times \times \mathbb{C}$ along their common open subset $U_{\{O\}} = (\mathbb{C}^\times)^2$.

(ii) Let $v_3 := -v_1 - v_2$ and $\sigma_1 := \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2$, $\sigma_2 := \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_3$, $\sigma_3 := \mathbb{R}_{\geq 0}v_2 + \mathbb{R}_{\geq 0}v_3$. then $\triangle = \{\sigma_1, \sigma_2, \sigma_3, \mathbb{R}_{\geq 0}v_1, \mathbb{R}_{\geq 0}v_2, \mathbb{R}_{\geq 0}v_3, \{O\}\}$ is a fan. We obtain $X(\triangle) = \mathbb{P}^2(\mathbb{C})$, the complex projective plane, by gluing these affine toric varieties along their common open subsets.

(iii) For $a \in \mathbb{Z}$, let $\sigma_1 := \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2$, $\sigma_2 := \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}(-v_2)$, $\sigma_3 := \mathbb{R}_{\geq 0}(-v_1 + av_2) + \mathbb{R}_{\geq 0}v_2$, $\sigma_4 := \mathbb{R}_{\geq 0}(-v_1 + av_2) + \mathbb{R}_{\geq 0}(-v_2)$. Then

$$\triangle = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \mathbb{R}_{\geq 0}v_1, \mathbb{R}_{\geq 0}v_2, \mathbb{R}_{\geq 0}(-v_1 + av_2), \mathbb{R}_{\geq 0}(-v_2), \{O\}\}$$

is a fan. We obtain the so-called *Hirzebruch surface* $X(\triangle) \cong \mathbb{P}^1 \times \mathbb{P}^1$, usually denoted by F_a.

Now let us see how the algebraic torus T_N acts on $X(\triangle)$. Let $g \in T_N$ and $x \in U_\sigma$, then $g : M \to \mathbb{C}^\times$ is a homomorphism and $x : S_\sigma \to \mathbb{C}$ satisfies $x(O) = 1$ and $x(u+u') = x(u)x(u')$ for $u, u' \in S_\sigma$. We can define $gx : S_\sigma \to \mathbb{C}$ by $(gx)(u) := g(u)x(u)$ for $u \in S_\sigma$. Note that gx is also an element of U_σ. This actually gives rise to an action of T_N on U_σ, hence on $X(\triangle)$ by natural gluing. In fact, we can decompose $X(\triangle)$ into T_N-orbits in terms of $\triangle$. For each $\sigma \in \triangle$ we can define a quotient algebraic torus of T_N as follows:

$$\mathrm{orb}(\sigma) := \{x : M \cap \sigma^\perp \to \mathbb{C}^\times \mid \text{group homomorphisms}\},$$

which can be regarded as a T_N-orbit in $X(\triangle)$. Conversely, every T_N-orbit is of this form and given in this way.

Proposition 1.14. *$\triangle$ is in one-to-one correspondence with the set of T_N-orbits in $X(\triangle)$. Furthermore,* $\mathrm{orb}(\sigma)$ *has a complementary dimension of σ in $N_\mathbb{R}$.*

We can easily see that $\mathrm{orb}(\{O\}) = U_{\{O\}} = T_N$ and $\mathrm{orb}(\sigma)$ is contained in the closure of $\mathrm{orb}(\tau)$ if and only if $\tau \prec \sigma$. Thus we have $U_\sigma = \bigcup_{\tau\prec\sigma} \mathrm{orb}(\tau)$.

As we already described in the preceding discussion, we can associate to $v \in N$ a one-parameter subgroup $\gamma_v : \mathbb{C}^\times \to T_N$ defined by $\gamma_v(t)(u) = t^{(u,v)}$. If we analyze the limit $\lim_{t\to 0} \gamma_v(t)$, we find that $\lim_{t\to 0} \gamma_v(t)$ exists in U_σ if and only if $v \in \sigma$. Following this way, we shall see that

Theorem 1.15. *A toric variety $X(\triangle)$ is compact if and only if its support $|\triangle|$ is the whole space $N_\mathbb{R}$.*

In this thesis, the lattice M is always given by a root lattice.

CHAPTER 2

Affine structures

In this chapter we construct a projective structure on a toric arrangement complement H°. This is equivalent to constructing an affine structure on $H^\circ \times \mathbb{C}^\times$, i.e., producing a torsion free flat connection on $H^\circ \times \mathbb{C}^\times$. In Section 2.1, we provide a general idea on how to construct such a desired connection on $M \times \mathbb{C}^\times$ out of a given connection on a complex manifold M. In Section 2.2, following the idea of last section, we do construct such a connection for $H^\circ \times \mathbb{C}^\times$, which is inspired by the work of Heckman and Opdam on special hypergeometric system associated with a root system. In Section 2.3, we show the constructed connection on $H^\circ \times \mathbb{C}^\times$ can be flat as long as we choose an appropriate bilinear form a^κ for it.

2.1. Projective structures

Let M be a complex manifold of dimension n.

Definition 2.1. A *projective structure* on M is given by an atlas of holomorphic charts for which the transition maps are projective-linear and which is maximal for that property. Likewise, an *affine structure* on M is given by an atlas of holomorphic charts for which the transition maps are affine-linear and which is maximal for that property.

So a projective structure on M is locally modelled on the pair $(\mathbb{P}^n, \mathrm{Aut}(\mathbb{P}^n))$ of projective space and projective group and an affine structure is locally modelled on the pair $(\mathbb{A}^n, \mathrm{Aut}(\mathbb{A}^n))$ of affine space and affine group.

We recall from [**6**] that an affine structure defines a subsheaf of rank $n+1$ in the structure sheaf $\mathcal{O}_M$ containing constants, the sheaf of locally affine-linear functions. The differentials of these make up a local system on the sheaf Ω_M of differentials on M, and such a local system is given by a holomorphic connection on Ω_M, $\nabla : \Omega_M \to \Omega_M \otimes \Omega_M$ with extension $\nabla : \Omega^k_M \otimes \Omega_M \to \Omega^{k+1}_M \otimes \Omega_M$ by the Leibniz rule $\nabla(\omega \otimes \zeta) = d(\omega) \otimes \zeta + (-1)^k \omega \wedge \nabla(\zeta)$ for $k \in \mathbb{N}$. This connection is flat and torsion free. For any connection ∇ on Ω_M its square $\nabla^2 : \Omega^k_M \otimes \Omega_M \to \Omega^{k+2}_M \otimes \Omega_M$ is a morphism of $\mathcal{O}_M$-modules, given by wedging with a section R of $\mathrm{End}_{\mathcal{O}_M}(\Omega_M, \Omega^2_M \otimes \Omega_M)$, called the curvature of ∇, and ∇ is flat if and only if $\mathrm{R} = 0$. The connection ∇ on Ω_M is also torsion free, which means that the composite of $\wedge : \Omega_M \otimes \Omega_M \to \Omega^2_M$ with $\nabla : \Omega_M \to \Omega_M \otimes \Omega_M$ is equal to the exterior derivative $d : \Omega_M \to \Omega^2_M$. Indeed flat differentials in Ω_M are then closed, and provide by the Poincaré lemma a subsheaf of $\mathcal{O}_M$ of rank $n+1$ containing constants. We refer to Deligne's lecture notes for an excellent exposition of the language of connections and more [**10**]. Conversely, a torsion free flat connection on the cotangent bundle of M defines an affine structure on M.

A projective structure can also be described in terms of a connection, at least locally. Let us first observe that such a structure on M defines locally a tautological $\mathbb{C}^\times$-bundle $\pi : L \to M$ whose total space has an affine structure and for which scalar multiplication respects the affine structure. This local $\mathbb{C}^\times$-bundle is unique up to scalar multiplication and need not be globally defined.

We write a projective structure on M in terms of an affine structure on $M \times \mathbb{C}^\times$ in the following lemma.

Proposition 2.2. *Let M be a complex manifold endowed with a holomorphic connection $\nabla : \Omega_M \to \Omega_M \otimes \Omega_M$ on its cotangent bundle. Suppose the connection is torsion free and that its curvature, viewed as a $\mathcal{O}_M$-homomorphism $\nabla\nabla$: $\Omega_M \to \Omega^2_M \otimes \Omega_M$, is given by wedging from the right with a symmetric section $-A$ of $\Omega_M \otimes \Omega_M : \zeta \mapsto -\zeta \wedge A$. Then an affine structure $\tilde{\nabla}$ on $M \times \mathbb{C}^\times$ is given by*

$$\begin{aligned}\tilde{\nabla}(\zeta) &= \nabla(\zeta) - \zeta \otimes \frac{dt}{t} - \frac{dt}{t} \otimes \zeta,\\ \tilde{\nabla}(\frac{dt}{t}) &= A - \frac{dt}{t} \otimes \frac{dt}{t},\end{aligned}$$

with $\zeta \in \Omega_M$ and t the coordinate on $\mathbb{C}^\times$. Moreover, its local affine functions are of the form $c + tf$, with $f \in \mathcal{O}_M$ satisfying $\nabla(df) + fA = 0$ and $c \in \mathbb{C}$ a constant.

Proof. Put $L := M \times \mathbb{C}^\times$ and denote by $\pi : L \to M$ and $t : L \to \mathbb{C}^\times$ the projections. Then we have

$$\Omega_L \cong \pi^*\Omega_M \oplus \mathcal{O}_L \frac{dt}{t}$$

and regard the natural map $\Omega_M \to \pi_*\pi^*\Omega_M$ as an inclusion. We have to verify that the connection $\tilde{\nabla} : \Omega_L \to \Omega_L \otimes \Omega_L$ defined above is flat and torsion free.

We first prove the flatness. Observe that for $\omega, \zeta \in \Omega_M$ we have

$$
\begin{aligned}
\tilde{\nabla}(\omega \otimes \zeta) &= d\omega \otimes \zeta - \omega \wedge (\nabla(\zeta) - \zeta \otimes \frac{dt}{t} - \frac{dt}{t} \otimes \zeta) \\
&= \nabla(\omega \otimes \zeta) + (\omega \wedge \zeta) \otimes \frac{dt}{t} - \frac{dt}{t} \wedge (\omega \otimes \zeta)
\end{aligned}
$$

which in turn implies

$$
\begin{aligned}
\tilde{\nabla}(\nabla(\zeta)) &= \nabla^2(\zeta) + (\wedge\nabla)(\zeta) \otimes \frac{dt}{t} - \frac{dt}{t} \wedge \nabla(\zeta) \\
&= \nabla^2(\zeta) + d\zeta \otimes \frac{dt}{t} - \frac{dt}{t} \wedge \nabla(\zeta)
\end{aligned}
$$

since ∇ is torsion free by assumption. Hence we get for $\zeta \in \Omega_M$

$$
\begin{aligned}
\tilde{\nabla}^2(\zeta) =& \tilde{\nabla}(\nabla(\zeta) - \zeta \otimes \frac{dt}{t} - \frac{dt}{t} \otimes \zeta) \\
=& \nabla^2(\zeta) + d\zeta \otimes \frac{dt}{t} - \frac{dt}{t} \wedge \nabla(\zeta) - d\zeta \otimes \frac{dt}{t} + \zeta \wedge (A - \frac{dt}{t} \otimes \frac{dt}{t}) \\
&+ \frac{dt}{t} \wedge (\nabla(\zeta) - \zeta \otimes \frac{dt}{t} - \frac{dt}{t} \otimes \zeta) \\
=& \nabla^2(\zeta) + \zeta \wedge A = 0
\end{aligned}
$$

by assumption. Observe that the above definition of $\tilde{\nabla}(dt/t)$ is equivalent to $\tilde{\nabla}(dt) = tA$, and so we get

$$
\begin{aligned}
\tilde{\nabla}^2(dt) &= \tilde{\nabla}(tA) = dt \wedge A + t\tilde{\nabla}(A) \\
&= dt \wedge A + t(\nabla(A) + (\wedge A) \otimes \frac{dt}{t} - \frac{dt}{t} \wedge A) \\
&= t\nabla(A) = 0
\end{aligned}
$$

because A is symmetric, and using the Bianchi identity $\nabla(A) = 0$. This proves that $\tilde{\nabla}$ is a flat connection.

The verification that $\tilde{\nabla}$ is torsion free is easy. Indeed for $\zeta \in \Omega_M$

$$
\begin{aligned}
&\wedge \tilde{\nabla}(\zeta) = \wedge\nabla(\zeta) - \zeta \wedge \frac{dt}{t} - \frac{dt}{t} \wedge \zeta = d\zeta \\
&\wedge \tilde{\nabla}(dt) = t(\wedge A) = 0
\end{aligned}
$$

as should.

Finally for $\varphi \in \mathcal{O}_L$ of the form $\sum f_k t^k$ with $f_k \in \mathcal{O}_M$ we get

$$\begin{aligned}
\tilde{\nabla}(d\varphi) &= \sum \tilde{\nabla}(t^k df_k + k f_k t^k \frac{dt}{t}) \\
&= \sum t^k (k\frac{dt}{t} \otimes df_k + \nabla(df_k) - df_k \otimes \frac{dt}{t} - \frac{dt}{t} \otimes df_k) \\
&\quad + \sum t^k (k df_k \otimes \frac{dt}{t} + k(k-1) f_k \frac{dt}{t} \otimes \frac{dt}{t} + k f_k A) \\
&= \sum t^k (\nabla(df_k) + k f_k A) + \sum k(k-1) t^k f_k \frac{dt}{t} \otimes \frac{dt}{t} \\
&\quad + \sum (k-1) t^k (df_k \otimes \frac{dt}{t} + \frac{dt}{t} \otimes df_k) = 0
\end{aligned}$$

if and only if $f_k = 0$ for $k \neq 0, 1$ and $f_0, f_1 \in \mathcal{O}_M$ are solutions of

$$df_0 = 0 \ , \ \nabla(df_1) + f_1 A = 0 \ .$$

This completes the proof of the lemma.

□

Given a projective structure on M the pair (∇, A) of a torsion free connection ∇ on Ω_M whose curvature is given by $\zeta \mapsto -\zeta \wedge A$ with A a symmetric section of $\Omega_M \otimes \Omega_M$ is not unique, because the proposition produces not just the tautological line bundle, but also a trivialization t. Let us see how this changes if we choose another local trivialization t'. Write $t' = te^g$, with $g \in \mathcal{O}_M$. From $\frac{dt'}{t'} = \frac{dt}{t} + dg$, we see that

$$\tilde{\nabla}(\zeta) = \nabla'(\zeta) - \zeta \otimes \frac{dt'}{t'} - \frac{dt'}{t'} \otimes \zeta$$

with

$$\nabla'(\zeta) := \nabla(\zeta) + dg \otimes \zeta + \zeta \otimes dg.$$

Furthermore,

$$\begin{aligned}
\tilde{\nabla}(\frac{dt'}{t'}) &= \tilde{\nabla}(\frac{dt}{t} + dg) \\
&= A - \frac{dt}{t} \otimes \frac{dt}{t} + \nabla(dg) - dg \otimes \frac{dt}{t} - \frac{dt}{t} \otimes dg \\
&= A + \nabla(dg) + dg \otimes dg - \frac{dt'}{t'} \otimes \frac{dt'}{t'} \\
&= A' - \frac{dt'}{t'} \otimes \frac{dt'}{t'}
\end{aligned}$$

with

$$A' := A + \nabla(dg) + dg \otimes dg.$$

It is worthwhile to write out the content of the above Proposition in local coordinates $z = (z^1, \cdots, z^n)$ on M. Let $\nabla^0 : \Omega_M \to \Omega_M \otimes \Omega_M$ be the connection defined by $\nabla^0(dz^k) = 0$ for all k.

Corollary 2.3. *In these local coordinates let* $\nabla = \nabla^0 + \Omega : \Omega_M \to \Omega_M \otimes \Omega_M$ *be a connection on* Ω_M*, so* $\Omega : \Omega_M \to \Omega_M \otimes \Omega_M$ *is a morphism of* $\mathcal{O}_M$*-modules and* $\Omega(dz^k) = \sum \Gamma_{ij}^k dz^i \otimes dz^j$ *with* Γ_{ij}^k *the connection coefficients of* ∇*. Let* A *be a quadratic differential on* M*, so* A *is a symmetric section of* $\Omega_M \otimes \Omega_M$ *given in the local coordinates as* $A = \sum A_{ij} dz^i \otimes dz^j$ *with* $A_{ij} = A_{ji}$ *for all* $1 \leq i, j \leq n$*. Then the linear system of second order differential equations* $\nabla(df) + fA = 0$ *for* $f \in \mathcal{O}_M$ *takes in these local coordinates the explicit form*

$$(\partial_i \partial_j + \sum \Gamma_{ij}^k \partial_k + A_{ij}) f = 0$$

for all $1 \leq i, j \leq n$*. It has local solution space of dimension at most* $n+1$ *with equality if and only if the connection* ∇ *is torsion free and its curvature* R *is given by* $\Omega_M \ni \zeta \mapsto -\zeta \wedge A \in \Omega_M^2 \otimes \Omega_M$*. In these local coordinates* ∇ *is torsion free if and only if* $\Gamma_{ij}^k = \Gamma_{ji}^k$ *for all* $1 \leq i, j, k \leq n$*, and* $\mathrm{R}(\zeta) = -\zeta \wedge A$ *for all* $\zeta \in \Omega_M$ *if and only if* $2\mathrm{R}_{lij}^k = \delta_j^k A_{il} - \delta_i^k A_{jl}$ *for all* $1 \leq i, j, k, l \leq n$ *with* δ *the Kronecker symbol and*

$$\mathrm{R}_{lij}^k = (\partial_i \Gamma_{lj}^k - \partial_j \Gamma_{li}^k) + \sum (\Gamma_{mi}^k \Gamma_{lj}^m - \Gamma_{mj}^k \Gamma_{li}^m)$$

the coefficients of the curvature matrix $\mathrm{R}_l^k = \sum \mathrm{R}_{lij}^k dz^i \wedge dz^j$ *of the curvature* R *in the basis* dz^l*.*

Proof. In these local coordinates we have $df = \sum (\partial_j f) dz^j$ for $f \in \mathcal{O}_M$ and so $\nabla^0(df) = \sum \partial_i \partial_j (f) dz^i \otimes dz^j$ and hence $\nabla(df) + fA = 0$ amounts to

$$\sum (\partial_i \partial_j f + \sum \Gamma_{ij}^k \partial_k f + A_{ij} f) dz^i \otimes dz^j = 0$$

which yields the above linear system of second differential equations. The connection ∇ is torsion free if and only if $\wedge \nabla = d$ which amounts to $\sum \Gamma_{ij}^k dz^i \wedge dz^j = 0$ or equivalently $\Gamma_{ij}^k = \Gamma_{ji}^k$ for all $1 \leq i, j, k \leq n$. The curvature R of ∇ sends dz^k to the element $\sum \mathrm{R}_{lij}^k (dz^i \wedge dz^j) \otimes dz^l$ and the condition that $\mathrm{R} = \nabla^2 : \Omega_M \to \Omega_M^2 \otimes \Omega_M$ is just equal to $\zeta \mapsto -\zeta \wedge A$ for $\zeta \in \Omega_M$ amounts to

$$\mathrm{R}(dz^k) = \sum \mathrm{R}_{lij}^k (dz^i \wedge dz^j) \otimes dz^l = \sum A_{il} (dz^i \wedge dz^k) \otimes dz^l = -dz^k \wedge A$$

for all $1 \leq k \leq n$ and so

$$\sum \mathrm{R}_{lij}^k (dz^i \wedge dz^j) = \sum A_{il} (dz^i \wedge dz^k)$$

for all $1 \le k, l \le n$. Contraction with the vector field ∂_m yields

$$\begin{aligned}\sum 2\mathrm{R}^k_{lmj} dz^j &= \sum \mathrm{R}^k_{lij}(\delta^i_m dz^j - \delta^j_m dz^i)\\ &= \sum A_{il}(\delta^i_m dz^k - \delta^k_m dz^i)\\ &= A_{ml} dz^k - \sum \delta^k_m A_{il} dz^i\\ &= \sum (\delta^k_j A_{ml} - \delta^k_m A_{jl}) dz^j\end{aligned}$$

for all $1 \le k, l, m \le n$. Hence the condition for the relation $\mathrm{R}(\zeta) = -\zeta \wedge A$ becomes

$$2\mathrm{R}^k_{lij} = \delta^k_j A_{il} - \delta^k_i A_{jl}$$

for all $1 \le i, j, k, l \le n$. □

2.2. Differential operators and connections on tori

We consider the situation discussed above in the special case where the underlying complex manifold is an algebraic torus.

Let $\mathfrak{a}$ be a real vector space of dimension n endowed with an inner product $(\cdot,\cdot)$ (making it a Euclidean vector space). The inner product identifies $\mathfrak{a}$ with its dual $\mathfrak{a}^*$, so that the latter also is endowed with an inner product, by abuse of notation still denoted by $(\cdot,\cdot)$. Suppose also given a reduced irreducible finite root system $R \subset \mathfrak{a}^*$. Then the corresponding orthogonal reflection for each $\alpha \in R$

$$s_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha, \quad \beta \in \mathfrak{a}^*$$

preserves the set R and the crystallographic condition

$$\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$$

holds for all $\alpha, \beta \in R$. Let $Q = \mathbb{Z}R$ denote the root lattice in $\mathfrak{a}^*$ and denote the corresponding dual root system in $\mathfrak{a}$ by $R^\vee$. We then have the coweight lattice $P^\vee = \mathrm{Hom}(Q, \mathbb{Z})$ of $R^\vee$ in $\mathfrak{a}$. Hence

$$H = \mathrm{Hom}(Q, \mathbb{C}^\times)$$

is a so-called adjoint algebraic torus with (rational) character lattice Q.

We denote the Lie algebra of H by $\mathfrak{h}$, so $\mathfrak{h} = \mathbb{C} \otimes \mathfrak{a}$ and $H = \mathfrak{h}/2\pi\sqrt{-1}P^\vee$ as a complex torus. For $v \in \mathfrak{h}$ we denote by ∂_v the associated translation invariant vector field on H. Likewise, if we are given $\phi \in \mathfrak{h}^*$, then we denote by $d\phi$ the associated translation invariant differential on H. In case ϕ determines a character of H (meaning $\phi \in Q$), we denote that character by e^ϕ. If $\exp : \mathfrak{h} \to H = \mathfrak{h}/2\pi\sqrt{-1}P^\vee$ is the exponential map with the inverse $\log : H \to \mathfrak{h}$, then we have $e^\phi(h) = e^{\phi(\log h)}$ for all $h \in H$. We also have $d\phi = (e^\phi)^*(\frac{dt}{t})$ with

t the coordinate on $\mathbb{C}^\times$. We denote the (flat) translation invariant connections on H and $H \times \mathbb{C}^\times$ by ∇^0 and $\tilde{\nabla}^0$ respectively (so that $\partial_v = \nabla^0_{\partial_v}$).

Each α in R determines a character e^α, then R generates the character lattice Q and each element of R is primitive in Q. So the set $R_+ := R/\pm$ of antipodal pairs in R indexes in one-one manner the kernels of these characters. The kernel $H_\alpha = \{h \in H \mid e^\alpha(h) = 1\}$ has its Lie algebra $\mathfrak{h}_\alpha$ which is the zero set of α. We call the finite collection of these hypertori H_α's a *toric arrangement* associated with a root system R. We write H° for the complement of the union of these hypertori as follows:

$$H^\circ := H - \cup_{\alpha \in R_+} H_\alpha.$$

Let K be the space of multiplicity parameters for R defined as the space of W-invariant functions

$$\kappa = (k_\alpha)_{\alpha \in R} \in \mathbb{C}^R$$

where W is the Weyl group generated by all reflections s_α. We shall sometimes write k_i instead of k_{α_i} if $\alpha_1, \cdots, \alpha_n$ is a basis of simple roots in R_+. It is clear that K is isomorphic to $\mathbb{C}^r$ as a $\mathbb{C}$-vector space if r is the number of W-orbits in R (i.e., $r = 1$ or 2). Hence for convenience, we sometimes also write k for k_1 and k' for k_n if $\alpha_n \notin W\alpha_1$ when no confusion can arise. But note that k' has a different meaning for type A_n, which can be seen from Remark 2.5.

We also have given for each $\alpha \in R$ a nonzero *coroot* $\alpha^\vee \in \mathfrak{h}$ such that $(-\alpha)^\vee = -\alpha^\vee$ and $\alpha^\vee(\alpha) = 2$. Let

$$a^\kappa : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}, \quad b^\kappa : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$$

be a symmetric bilinear form and a symmetric bilinear map respectively, which are invariant and equivariant under the W action respectively. We notice that a^κ is just a multiple of the given inner product $(\cdot, \cdot)$ by the Schur's lemma since R is irreducible.

Lemma 2.4. *If R is irreducible then b^κ vanishes unless R is of type A_n for $n \geq 2$ in which case there exists a $k' \in \mathbb{C}$ such that*

$$b^\kappa(u, v) = \frac{1}{2} k' \sum_{\alpha > 0} \alpha(u)\alpha(v)\alpha' \quad \textit{for any } u, v \in \mathfrak{h}$$

with $\alpha' = \varepsilon_i + \varepsilon_j - \frac{2}{n+1}\sum_l \varepsilon_l$ if we take the construction of α from Bourbaki [**2**]*: $\alpha = \varepsilon_i - \varepsilon_j$ for $1 \leq i < j \leq n+1$.*

Proof. We write b for b^κ if no confusion arises. Obviously we can identify b with an element of $\mathrm{Hom}(\mathfrak{a}, \mathrm{Sym}^2(\mathfrak{a}^*))^W$. First fix a positive definite generator g of $\mathrm{Sym}^2(\mathfrak{a}^*)^W$ and then choose a line $L \subset \mathfrak{a}$ such that its g-orthogonal complement $H \subset \mathfrak{a}$ is a hyperplane for which $R_H := R \cap H$ spans H and is an irreducible root system. We decompose $\mathrm{Sym}^2(\mathfrak{a}^*) = \mathrm{Sym}^2(H^*) \oplus (L^* \otimes H^*) \oplus (H^* \otimes L^*) \oplus (L^*)^{\otimes 2}$. If we consider the $W(R_H)$-invariant part, the middle two summands immediately become trivial since $(H^*)^{W(R_H)} = 0$. Then we have $\mathrm{Sym}^2(\mathfrak{a}^*)^{W(R_H)} = \mathbb{R}g_L \oplus \mathbb{R}g$ where $g = g_H + g_L$ and g_H resp. g_L is the restriction of g on H resp. L.

Let $f \in \mathrm{Hom}(\mathfrak{a}, \mathrm{Sym}^2(\mathfrak{a}^*))^W$, then $f(v) = \mu g_L + \lambda g$ for some $v \in L$ since L belongs to the $W(R_H)$-invariant part. Assume that there exists a w such that $w(v) = -v$. Since w preserves both g_L and g, we must have $\mu = \lambda = 0$ by the linearity of f and thus $f(v) = 0$. Since the W-orbit of v spans V, it follows that $f = 0$.

This assumption is certainly satisfied when $-1 \in W$. Let's consider the remaining cases: $A_{n\geq 2}$, D_{odd} and E_6. For E_6, we take v to be a root, then R_H is of type A_5. For D_{odd}, we take v perpendicular to a subsystem of type D_{n-1}, then there is a w whose restriction to H is a reflection (in terms of the construction in Bourbaki: $v = \varepsilon_1$ and $w = s_{\varepsilon_1 - \varepsilon_2} s_{\varepsilon_1 + \varepsilon_2}$).

When R is of type $A_{n\geq 2}$, we use the construction in Bourbaki again: $\mathfrak{a}$ is the hyperplane in $\mathbb{R}^{n+1}$ defined by $\sum_{i=1}^{n+1} x_i = 0$. Put $\bar{x}_i := x_i|\mathfrak{a}$ so that $\sum_i \bar{x}_i = 0$. Let $\bar{\varepsilon}_i \in \mathfrak{a}$ be the orthogonal projection of $\varepsilon_i \in \mathbb{R}^{n+1}$ in $\mathfrak{a}$. The orthogonal complement of ε_i in $\mathfrak{a}$ is spanned by a subsystem of type A_{n-1}. Note that all the $\bar{\varepsilon}_i$'s make up a W-orbit with sum zero. So if we write $f(\bar{\varepsilon}_i) = \mu\bar{x}_i^2 + \lambda g$, sum them up, we get $0 = \sum_{i=1}^{n+1} f(\bar{\varepsilon}_i) = \mu \sum_{i=0}^{n+1} \bar{x}_i^2 + (n+1)\lambda g$. Hence we have $f(\bar{\varepsilon}_i) = \mu(\bar{x}_i^2 - \frac{1}{n+1}\sum_{i=1}^{n+1} \bar{x}_i^2)$. This indeed defines an element of $\mathrm{Hom}(\mathfrak{a}, \mathrm{Sym}^2(\mathfrak{a}^*))^W$ and we thus have $\dim(\mathrm{Hom}(\mathfrak{a}, \mathrm{Sym}^2(\mathfrak{a}^*))^W) = 1$.

Let $b_0(u,v) = \sum_{\alpha>0} \alpha(u)\alpha(v)\alpha'$. Since $w(\alpha') = w(\alpha)'$ we have $wb_0(u,v) = b_0(wu, wv)$ for all $u, v \in \mathfrak{h}$ and $w \in W(A_n) = \mathfrak{S}_{n+1}$. Then we see that b_0 is a generator of $\mathrm{Hom}(\mathrm{Sym}^2\mathfrak{h}, \mathfrak{h})^W$. □

Remark 2.5. In fact, for type A_n, another generator is obtained by taking $v \in \mathfrak{a} \mapsto \partial_v\sigma_3|\mathfrak{a}$ where $\bar{\sigma}_3 := \sigma_3|\mathfrak{a}$ is an element of $(\mathrm{Sym}^3(\mathfrak{a}^*))^W$. This point will become more clear when we discuss the toric Lauricella case in Example 2.12.

And because b^κ exists for type A_n, we would like to include k' in κ for type A_n.

Then we define the vector fields

$$X_\alpha := \frac{e^\alpha + 1}{e^\alpha - 1}\partial_{\alpha^\vee}$$

(notice that $X_{-\alpha} = X_\alpha$) and consider for $u, v \in \mathfrak{h}$, the second order differential operator on $\mathcal{O}_{H^\circ}$ defined by

$$D^\kappa_{u,v} := \partial_u\partial_v + \frac{1}{2}\sum_{\alpha>0} k_\alpha \alpha(u)\alpha(v)X_\alpha + \partial_{b^\kappa(u,v)} + a^\kappa(u,v).$$

We want this system to define a projective structure on H°. That means for each multiplicity parameter κ and each equivariant bilinear map b^κ as above there exists a W-invariant bilinear form a^κ such that the system of differential equations $D^\kappa_{u,v} f = 0$ for all $u, v \in \mathfrak{h}$ is integrable. It is obvious that this projective structure is invariant under the action of W.

Taking the cue from Proposition 2.2, we associate to these data connections $\nabla^\kappa = \nabla^0 + \Omega^\kappa$ and $\tilde{\nabla}^\kappa = \tilde{\nabla}^0 + \tilde{\Omega}^\kappa$ on the cotangent bundles of H° and $H^\circ \times \mathbb{C}^\times$ with $\Omega^\kappa \in \mathrm{Hom}(\Omega_{H^\circ}, \Omega_{H^\circ} \otimes \Omega_{H^\circ})$ given by

$$\Omega^\kappa : \zeta \in \Omega_{H^\circ} \mapsto \frac{1}{2} \sum_{\alpha>0} k_\alpha \zeta(X_\alpha) d\alpha \otimes d\alpha + (B^\kappa)^*(\zeta) \tag{2.1}$$

and $\tilde{\Omega}^\kappa \in \mathrm{Hom}(\Omega_{H^\circ \times \mathbb{C}^\times}, \Omega_{H^\circ \times \mathbb{C}^\times} \otimes \Omega_{H^\circ \times \mathbb{C}^\times})$ given by

$$\tilde{\Omega}^\kappa : \begin{cases} \zeta \in \Omega_{H^\circ} \mapsto \frac{1}{2} \sum_{\alpha>0} k_\alpha \zeta(X_\alpha) d\alpha \otimes d\alpha + (B^\kappa)^*(\zeta) - \zeta \otimes \frac{dt}{t} - \frac{dt}{t} \otimes \zeta, \\ \frac{dt}{t} \in \Omega_{\mathbb{C}^\times} \mapsto A^\kappa - \frac{dt}{t} \otimes \frac{dt}{t}. \end{cases} \tag{2.2}$$

Here t is the coordinate for $\mathbb{C}^\times$ and A^κ and B^κ denote the translation invariant tensor fields on H or $H \times \mathbb{C}^\times$ defined by a^κ and b^κ respectively. According to (2.1), (2.2), we can write Ω^κ and $\tilde{\Omega}^\kappa$ explicitly:

$$\Omega^\kappa := \frac{1}{2} \sum_{\alpha>0} k_\alpha d\alpha \otimes d\alpha \otimes X_\alpha + (B^\kappa)^*,$$

$$\begin{aligned} \tilde{\Omega}^\kappa := & \frac{1}{2} \sum_{\alpha>0} k_\alpha d\alpha \otimes d\alpha \otimes X_\alpha + (B^\kappa)^* + c^\kappa \sum_{\alpha>0} d\alpha \otimes d\alpha \otimes t\frac{\partial}{\partial t} \\ & - \sum_{\alpha_i \in \mathfrak{B}} d\alpha_i \otimes \frac{dt}{t} \otimes \partial_{p_i} - \frac{dt}{t} \otimes \frac{dt}{t} \otimes t\frac{\partial}{\partial t} - \sum_{\alpha_i \in \mathfrak{B}} \frac{dt}{t} \otimes d\alpha_i \otimes \partial_{p_i}. \end{aligned}$$

Here c^κ is a constant for each κ such that $A^\kappa = c^\kappa \sum_{\alpha>0} d\alpha \otimes d\alpha$, $\mathfrak{B}$ is a fundamental system for R and p_i is the dual basis of $\mathfrak{h}$ to α_i such that $\alpha_i(p_j) = \delta^i_j$ where δ^i_j is the Kronecker delta.

Example 2.6. We take a root system of type A_2. We compute the curvature form of the connection defined by this root system. For $\alpha, \beta, \gamma \in R_+$, we write

$$\Omega := \frac{e^\alpha + 1}{e^\alpha - 1} d\alpha \otimes d\alpha \otimes \partial_{\alpha^\vee} + \frac{e^\beta + 1}{e^\beta - 1} d\beta \otimes d\beta \otimes \partial_{\beta^\vee} + \frac{e^\gamma + 1}{e^\gamma - 1} d\gamma \otimes d\gamma \otimes \partial_{\gamma^\vee}.$$

$$\nabla = \nabla^0 + \Omega \text{ such that } \nabla^0(d\alpha) = 0.$$

Here we let $k_\alpha = 2$ for all α and $k' = 0$.

Let $\zeta = c_1 d\alpha + c_2 d\beta \in \Omega_{H^\circ}$, then we have

$$\nabla(\zeta) = \frac{e^\alpha + 1}{e^\alpha - 1} d\alpha \otimes \zeta(\partial_{\alpha^\vee}) d\alpha + \frac{e^\beta + 1}{e^\beta - 1} d\beta \otimes \zeta(\partial_{\beta^\vee}) d\beta + \frac{e^\gamma + 1}{e^\gamma - 1} d\gamma \otimes \zeta(\partial_{\gamma^\vee}) d\gamma,$$

and furthermore,

$$
\begin{aligned}
&\nabla\nabla(\zeta)\\
=&-\frac{e^{\alpha}+1}{e^{\alpha}-1}d\alpha\wedge\nabla(\zeta(\partial_{\alpha^\vee})d\alpha)-\frac{e^{\beta}+1}{e^{\beta}-1}d\beta\wedge\nabla(\zeta(\partial_{\beta^\vee})d\beta)\\
&-\frac{e^{\gamma}+1}{e^{\gamma}-1}d\gamma\wedge\nabla(\zeta(\partial_{\gamma^\vee})d\gamma)\\
=&-\frac{e^{\alpha}+1}{e^{\alpha}-1}d\alpha\wedge\zeta(\partial_{\alpha^\vee})\Big(\frac{e^{\alpha}+1}{e^{\alpha}-1}d\alpha\otimes\partial_{\alpha^\vee}(d\alpha)d\alpha+\frac{e^{\beta}+1}{e^{\beta}-1}d\beta\otimes\partial_{\beta^\vee}(d\alpha)d\beta\\
&+\frac{e^{\gamma}+1}{e^{\gamma}-1}d\gamma\otimes\partial_{\gamma^\vee}(d\alpha)d\gamma\Big)\\
&-\frac{e^{\beta}+1}{e^{\beta}-1}d\beta\wedge\zeta(\partial_{\beta^\vee})\Big(\frac{e^{\alpha}+1}{e^{\alpha}-1}d\alpha\otimes\partial_{\alpha^\vee}(d\beta)d\alpha+\frac{e^{\beta}+1}{e^{\beta}-1}d\beta\otimes\partial_{\beta^\vee}(d\beta)d\beta\\
&+\frac{e^{\gamma}+1}{e^{\gamma}-1}d\gamma\otimes\partial_{\gamma^\vee}(d\beta)d\gamma\Big)\\
&-\frac{e^{\gamma}+1}{e^{\gamma}-1}d\gamma\wedge\zeta(\partial_{\gamma^\vee})\Big(\frac{e^{\alpha}+1}{e^{\alpha}-1}d\alpha\otimes\partial_{\alpha^\vee}(d\gamma)d\alpha+\frac{e^{\beta}+1}{e^{\beta}-1}d\beta\otimes\partial_{\beta^\vee}(d\gamma)d\beta\\
&+\frac{e^{\gamma}+1}{e^{\gamma}-1}d\gamma\otimes\partial_{\gamma^\vee}(d\gamma)d\gamma\Big),
\end{aligned}
$$

then making use of $\alpha+\beta+\gamma=0$ and $d\alpha\wedge d\beta=d\beta\wedge d\gamma=d\gamma\wedge d\alpha$, we can write the curvature form as follows,

$$
\begin{aligned}
\nabla\nabla=&-\frac{e^{\alpha}+1}{e^{\alpha}-1}\frac{e^{\beta}+1}{e^{\beta}-1}d\alpha\wedge d\beta\otimes(d\alpha\otimes\partial_{\beta^\vee}-d\beta\otimes\partial_{\alpha^\vee})\\
&-\frac{e^{\gamma}+1}{e^{\gamma}-1}\frac{e^{\alpha}+1}{e^{\alpha}-1}d\gamma\wedge d\alpha\otimes(d\gamma\otimes\partial_{\alpha^\vee}-d\alpha\otimes\partial_{\gamma^\vee})\\
&-\frac{e^{\beta}+1}{e^{\beta}-1}\frac{e^{\gamma}+1}{e^{\gamma}-1}d\beta\wedge d\gamma\otimes(d\beta\otimes\partial_{\gamma^\vee}-d\gamma\otimes\partial_{\beta^\vee})\\
=&-d\alpha\wedge d\beta\otimes(d\beta\otimes\partial_{\alpha^\vee}-d\alpha\otimes\partial_{\beta^\vee}).
\end{aligned}
$$

Then let

$$
\begin{aligned}
A&=d\alpha\otimes d\alpha+d\beta\otimes d\beta+d\gamma\otimes d\gamma\\
&=2d\alpha\otimes d\alpha+2d\beta\otimes d\beta+d\alpha\otimes d\beta+d\beta\otimes d\alpha,
\end{aligned}
$$

we can easily verify that

$$\nabla\nabla(\zeta)=-\zeta\wedge A.$$

Since Ω^κ and $\tilde{\Omega}^\kappa$ take values in the symmetric tensors, the connections they define are torsion free. The inversion involution of $H^\circ\times\mathbb{C}^\times$ acts on its space of logarithmic differentials, so that the latter decomposes into its subspace of invariants and the subspace of anti-invariants. The collection

$\{\frac{e^\alpha+1}{e^\alpha-1}d\alpha\}_{\alpha\in R_+}$ consists of linearly independent invariants, whereas the anti-invariants are the translation invariant differentials $\{d\alpha\}_{\alpha\in R_+}$. In particular, $\tilde{\Omega}^\kappa$ is logarithmic (in the sense that its matrix entries are logarithmic differentials) and is uniquely given by the form (2.2). An associated connection ∇^κ on the cotangent bundle of H° is given by the corresponding connection matrix Ω^κ. We have from Corollary 2.3:

Lemma 2.7. *Let $f \in \mathcal{O}_{H^\circ}$. Then we have $D^\kappa_{u,v}(f) = 0$ for all $u, v \in \mathfrak{h}$ if and only if the function $\tilde{f}(z,t) := c + f(z)t$ has a flat differential relative to $\tilde{\nabla}^\kappa$. Moreover, any exact differential in $\Omega_{H^\circ\times\mathbb{C}^\times}$ which is $\tilde{\nabla}^\kappa$-flat is of the form $d\tilde{f}$. Then the connection $\tilde{\nabla}^\kappa$ defines an affine structure on $H^\circ \times \mathbb{C}^\times$ and hence ∇^κ defines a projective structure on H° if and only if $-A^\kappa$ represents the curvature of ∇^κ.*

Then we can rewrite the above lemma in the form of a special hypergeometric system associated with a root system R which is due to [**7**].

Theorem 2.8. *Let $f \in \mathcal{O}_{H^\circ}$. The system of $n(n+1)/2$ linearly independent differential equations*

$$(\partial_u\partial_v + \frac{1}{2}\sum_{\alpha>0} k_\alpha \alpha(u)\alpha(v)\frac{e^\alpha(h)+1}{e^\alpha(h)-1}\partial_{\alpha^\vee} + \partial_{b^\kappa(u,v)} + a^\kappa(u,v))f(h) = 0 \;\forall u,v \in \mathfrak{h} \tag{2.3}$$

is an integrable system on H° if and only if the function $\tilde{f}(z,t) := c + f(z)t$ has a flat differential relative to $\tilde{\nabla}^\kappa$ and $-A^\kappa$ represents the curvature of ∇^κ, where A^κ is the translation invariant tensor field on H° defined by a^κ.

Proof. From Proposition 2.2, we can know that the function $\tilde{f}(z,t) := c+f(z)t$ has a flat differential relative to $\tilde{\nabla}^\kappa$ is equivalent to $f \in \mathcal{O}_{H^\circ}$ satisfying $\nabla^\kappa(df) + fA^\kappa = 0$. While the integrability of the system can be guaranteed by that $-A^\kappa$ represents the curvature of ∇^κ.

For $\omega \in \Omega^1(H^\circ)$, we have

$$\nabla^\kappa(\omega) = d\omega + \frac{1}{2}\sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1} d\alpha \otimes d\alpha \cdot \alpha^\vee(\omega) + (B^\kappa)^*(\omega).$$

Its covariant derivative in direction $v \in \mathfrak{h}$ is:

$$\nabla^\kappa_v(\omega) = \partial_v\omega + \frac{1}{2}\sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1} d\alpha \cdot \alpha(v)\alpha^\vee(\omega) + (B^\kappa_v)^*(\omega).$$

Let $\omega = df$, we have

$$\begin{aligned}\nabla^\kappa_v(df) &= \partial_v(df) + \frac{1}{2}\sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1} d\alpha \cdot \alpha(v)\alpha^\vee(df) + (B^\kappa_v)^*(df)\\ &= d(\partial_v f) + \frac{1}{2}\sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1} \alpha(v)\cdot\partial_{\alpha^\vee} f \cdot d\alpha + (B^\kappa_v)^*(df).\end{aligned}$$

Now contraction with u:

$$\nabla^\kappa_v(df)(u) = \partial_u\partial_v f + \frac{1}{2}\sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1}\alpha(u)\alpha(v)\partial_{\alpha^\vee} f + \partial_{b^\kappa(u,v)} f,$$

yields an element of $\mathcal{O}_{H^\circ}$.

So, $\nabla^\kappa(df) + fA^\kappa = 0$ is equivalent to

$$\partial_u\partial_v f + \frac{1}{2}\sum_{\alpha>0} k_\alpha\alpha(u)\alpha(v)\frac{e^\alpha+1}{e^\alpha-1}\partial_{\alpha^\vee} f + \partial_{b^\kappa(u,v)} f + a^\kappa(u,v)f = 0 \ \forall u, v \in \mathfrak{h}$$

$\square$

We call (2.3) the special hypergeometric system with multiplicity parameter κ.

2.3. Flatness of $\tilde{\nabla}^\kappa$

We want to see whether or not $\tilde{\nabla}^\kappa$ is indeed flat. For this we wish to apply the flatness criterion (1.2) of [**21**] to $\tilde{\nabla}^\kappa$. Here we restate the criterion as follows.

Lemma 2.9. *Let U be a connected complex manifold. Suppose that $\overline{U} \supset U$ is a smooth completion of U which adds to U an arrangement-like divisor D whose irreducible components D_i are smooth. Suppose that $\overline{U}$ has no nonzero regular 2-forms and that any irreducible component D_i has no nonzero regular 1-forms. Then a logarithmic connection E on the trivial vector bundle $U \times V$ over U is flat if and only if for every intersection I of two distinct irreducible components of D, the sum $\sum_{D_i\supset I} \mathrm{Res}_{D_i} E$ commutes with each of its terms $\mathrm{Res}_{D_i} E$ ($D_i \supset I$).*

Proof. First we prove the following fact.

Assertion: *The condition that $[\sum_{D_i\supset I} \mathrm{Res}_{D_i} E, \mathrm{Res}_{D_i} E] = 0$ is equivalent to $\mathrm{Res}_I \mathrm{Res}_{D_i} \mathrm{R}(\nabla) = 0$.*

The connection E on U could be locally written as

$$E = \sum_i f_i \frac{dl_i}{l_i} \otimes E_i + \sum_i \omega_i \otimes E'_i,$$

where f_i's are holomorphic functions, D_i is given by $l_i = 0$, ω_i's are holomorphic 1-forms and E_i's, E_i''s are the endomorphisms of V. Then we have

$$\mathrm{Res}_{D_i} E = f_i|_{l_i=0} E_i$$

and

$$\begin{aligned} E \wedge E =& \sum_{i,j} f_i f_j \frac{dl_i}{l_i}\frac{dl_j}{l_j} \otimes E_i E_j + \sum_{i,j} f_i \frac{dl_i}{l_i} \wedge \omega_j \otimes (E_i E_j' - E_j' E_i) \\ &+ \sum_{i,j} \omega_i \omega_j \otimes E_i' E_j' \\ =& \sum_{i<j} f_i f_j \frac{dl_i}{l_i}\frac{dl_j}{l_j} \otimes (E_i E_j - E_j E_i) + \sum_{i,j} f_i \frac{dl_i}{l_i} \wedge \omega_j \otimes (E_i E_j' - E_j' E_i) \\ &+ \sum_{i,j} \omega_i \omega_j \otimes E_i' E_j'. \end{aligned}$$

Then

$$\mathrm{Res}_{D_i} E \wedge E = \sum_{j:j\neq i} f_i f_j \frac{dl_j}{l_j}|_{l_i=0} \otimes (E_i E_j - E_j E_i) + \sum_{j:j\neq i} f_i \omega_j \otimes [E_i, E_j'],$$

As $I \subset D_i$ is given by $D_j \cap D_i$ for any $j \neq i$ with $D_j \supset I$, we have

$$\mathrm{Res}_I \mathrm{Res}_{D_i} E \wedge E = \sum_{j:D_j \supset I, j\neq i} f_i f_j|_I (E_i E_j - E_j E_i) =$$

$$\sum_{j:D_j \supset I, j\neq i} f_i f_j|_I [E_i, E_j] = [f_i|_I E_i, \sum_j f_j|_I E_j] = [\mathrm{Res}_{D_i} E, \sum \mathrm{Res}_{D_j} E].$$

Since the double residue of dE is obviously zero (any term of dE is of at most simple pole), the assertion follows.

Let's continue to prove the lemma. Necessity is obvious, but it is also sufficient: If the double residue of $\mathrm{R}(\nabla)$ is equal to zero, then $\mathrm{Res}_{D_i}\mathrm{R}(\nabla)$ has no pole along $I \subset D_j \cap D_i$ for $\forall D_j \neq D_i$, hence $\mathrm{Res}_{D_i}\mathrm{R}(\nabla)$ has as coefficients regular 1-form along D_i, but there is no nonzero regular 1-form along D_i, we then have $\mathrm{Res}_{D_i}\mathrm{R}(\nabla) = 0$. Again, $\mathrm{R}(\nabla)$ has no pole along D_i, hence $\mathrm{R}(\nabla)$ has as coefficients regular 2-form everywhere, but there is no nonzero regular 2-form on $\overline{U}$, we then have $\mathrm{R}(\nabla) = 0$. □

From the lemma above, we can see that it requires a compactification of $H^\circ \times \mathbb{C}^\times$ with a boundary which is arrangementlike in order to invoke the flatness criterion. We shall take this to be of the form $\hat{H}_\Sigma \times \mathbb{P}^1$, where the first factor is defined below.

Recall that H has a unique $\mathbb{Q}$-structure which is split, i.e., for which $H(\mathbb{Q})$ is isomorphic to a product of copies of $\mathbb{Q}^\times$. Each homomorphism $u : \mathbb{C}^\times \to H$

defines a tangent vector $Du(t\frac{\partial}{\partial t}) \in \mathfrak{h}(\mathbb{Q})$ and these tangent vectors span a lattice $\check{X}(H) \subset \mathfrak{h}(\mathbb{Q})$, called the *cocharacter lattice*. We shall identify $\check{X}(H)$ with $\mathrm{Hom}(\mathbb{C}^\times, H) = P^\vee$.

Since the $\mathfrak{h}_\alpha$'s are defined over $\mathbb{Q}$, they cut up $\mathfrak{h}(\mathbb{R})$ according to a rational cone decomposition Σ. The latter determines a compact torus embedding $H \subset \hat{H}$ whose boundary is a union of toric divisors, indexed by the one-dimensional faces of Σ. We denote by Π the set of primitive elements of $\check{X}(H)$ in spanning a face of Σ and associate an element p of Π with a boundary divisor D_p at the place of infinity. Our assumption that e^α is primitive in $X(H) = Q$ implies that H_α is connected and hence irreducible. So an irreducible component of $\hat{H} - H^\circ$ is now either the closure $\hat{H}_\alpha$ in $\hat{H}$ of some H_α or is equal to some D_p with $p \in \Pi$.

Two distinct boundary divisors meet precisely when the corresponding one dimensional faces of Σ span a two dimensional face. Clearly, the divisors $(t = 0)$ and $(t = \infty)$ meet all other divisors, and $\hat{H}_\alpha$ meets D_p if and only if $\alpha(p) = 0$.

We shall not make any notational distinction between a connection on the cotangent bundle of $H^\circ \times \mathbb{C}^\times$ and the associated one on its tangent bundle, i.e., the connection on its tangent bundle is also denoted by $\tilde{\nabla}^\kappa$. In fact, the associated (dual) connection on the tangent bundle of $H^\circ \times \mathbb{C}^\times$ is characterized by the property that the pairing between vector fields and differentials is flat. So its connection form is $-(\tilde{\Omega}^\kappa)^*$.

The residue of $(\tilde{\Omega}^\kappa)^*$ along these divisors are as follows: define elements of $\mathrm{End}(\mathfrak{h}) \subset \mathrm{End}(\mathfrak{h} \oplus \mathbb{C})$ by

$$\begin{aligned} u_\alpha &:= k_\alpha(\alpha^\vee \otimes \alpha), \\ U_x &:= -\frac{1}{4}\sum_{\alpha \in R} |\alpha(x)| u_\alpha, \quad x \in \mathfrak{h}(\mathbb{R}). \end{aligned}$$

So $u_{-\alpha} = u_\alpha$ and $U_{-x} = U_x$. Notice the dependence of U_x on x is piecewise linear (relative to Σ) and continuous. For $z \in \mathfrak{h}$, we define $b_z^\kappa \in \mathrm{End}(\mathfrak{h})$ and $a_z^\kappa \in \mathfrak{h}^*$ as follows:

$$\begin{aligned} b_z^\kappa(w) &:= b^\kappa(z, w), \\ a_z^\kappa(w) &:= a^\kappa(z, w). \end{aligned}$$

We first need to compute the following residues.

Notice that $d\alpha = d(\log e^\alpha) = \frac{de^\alpha}{e^\alpha}$ and the mappings:

$$\mathbb{C}^\times \xrightarrow{\gamma_p} H \xrightarrow{e^\alpha} \mathbb{C}^\times$$
$$t \mapsto t^{\alpha(p)},$$

we have

$$\begin{aligned}\mathrm{Res}_{\hat{H}_\alpha\times\mathbb{P}^1}\frac{e^\alpha+1}{e^\alpha-1}d\alpha &= \mathrm{Res}_{(e^\alpha=1)}\frac{e^\alpha+1}{e^\alpha-1}\frac{d(e^\alpha-1)}{e^\alpha}\\ &= 2,\end{aligned}$$

$$\begin{aligned}\mathrm{Res}_{D_p\times\mathbb{P}^1}\frac{e^\alpha+1}{e^\alpha-1}d\alpha &= \mathrm{Res}_{t=0}\gamma_p^*(\frac{e^\alpha+1}{e^\alpha-1}\frac{de^\alpha}{e^\alpha})\\ &= \mathrm{Res}_{t=0}\frac{t^{\alpha(p)}+1}{t^{\alpha(p)}-1}\alpha(p)\frac{dt}{t}\\ &= \begin{cases}\frac{t^{\alpha(p)}+1}{t^{\alpha(p)}-1}\alpha(p)\frac{dt}{t} = -\alpha(p) & \text{if} \quad \alpha(p)>0\\ \frac{1+t^{-\alpha(p)}}{1-t^{-\alpha(p)}}\alpha(p)\frac{dt}{t} = +\alpha(p) & \text{if} \quad \alpha(p)<0\end{cases}\\ &= -|\alpha(p)|,\end{aligned}$$

$$\begin{aligned}\mathrm{Res}_{D_p\times\mathbb{P}^1}d\alpha &= \mathrm{Res}_{t=0}\gamma_p^*(\frac{de^\alpha}{e^\alpha})\\ &= \mathrm{Res}_{t=0}\alpha(p)\frac{dt}{t}\\ &= \alpha(p);\end{aligned}$$

then we can compute

$$\begin{aligned}\mathrm{Res}_{\hat{H}_\alpha\times\mathbb{P}^1}(\tilde{\Omega}^\kappa)^* &= \frac{1}{2}\mathrm{Res}_{(e^\alpha=1)}k_\alpha\frac{e^\alpha+1}{e^\alpha-1}d\alpha\cdot\alpha^\vee\otimes\alpha\\ &= k_\alpha(\alpha^\vee\otimes\alpha)\\ &= u_\alpha,\end{aligned}$$

$$
\begin{aligned}
&\mathrm{Res}_{D_p\times\mathbb{P}^1}(\tilde{\Omega}^\kappa)^* \\
&\quad=\frac{1}{4}\sum_{\alpha\in R}\mathrm{Res}_{D_p\times\mathbb{P}^1}k_\alpha\frac{e^\alpha+1}{e^\alpha-1}d\alpha\cdot\alpha^\vee\otimes\alpha+\mathrm{Res}_{D_p\times\mathbb{P}^1}B^\kappa \\
&\qquad+c^\kappa\sum_{\alpha>0}\mathrm{Res}_{D_p\times\mathbb{P}^1}d\alpha\cdot t\frac{\partial}{\partial t}\otimes\alpha-\sum_{\alpha_i\in\mathfrak{B}}\mathrm{Res}_{D_p\times\mathbb{P}^1}d\alpha_i\cdot p_i\otimes\frac{dt}{t} \\
&\quad=\frac{1}{4}\sum_{\alpha\in R}k_\alpha(-|\alpha(p)|\alpha^\vee\otimes\alpha)+\mathrm{Res}_{D_p\times\mathbb{P}^1}B^\kappa \\
&\qquad+c^\kappa\sum_{\alpha>0}\alpha(p)\cdot t\frac{\partial}{\partial t}\otimes\alpha-\sum_{\alpha_i\in\mathfrak{B}}\alpha_i(p)\cdot p_i\otimes\frac{dt}{t} \\
&\quad=U_p+b_p^\kappa+t\frac{\partial}{\partial t}\otimes a_p^\kappa-p\otimes\frac{dt}{t},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Res}_{t=0}(\tilde{\Omega}^\kappa)^* &= \mathrm{Res}_{t=0}(-\frac{dt}{t})\cdot t\frac{\partial}{\partial t}\otimes\frac{dt}{t}+\sum_{\alpha_i\in\mathfrak{B}}\mathrm{Res}_{t=0}(-\frac{dt}{t})\cdot p_i\otimes\alpha_i \\
&= -t\frac{\partial}{\partial t}\otimes\frac{dt}{t}-\sum_{\alpha_i\in\mathfrak{B}}p_i\otimes\alpha_i \\
&= -1_{\mathbb{C}}-1_{\mathfrak{h}} \\
&= -1_{\mathfrak{h}\oplus\mathbb{C}} \\
&= -\mathrm{Res}_{t=\infty}(\tilde{\Omega}^\kappa)^*.
\end{aligned}
$$

We shall sometimes drop κ from ∇^κ, $\tilde{\nabla}^\kappa$, a^κ and b^κ when no confusion arises, but we need to bear in mind all these notations appearing in what follows in this Section depend on κ unless other specified.

Having these residues on hand and making use of Lemma 2.9, we have the flatness criterion for $\tilde{\nabla}$ as follows.

Lemma 2.10. *The connection $\tilde{\nabla}$ is flat (and thus defines an affine structure on $H^\circ\times\mathbb{C}^\times$) if and only if the following conditions hold:*

(1) for every rank two sublattice $L\subset Q$, the sum $\sum_{\alpha\in R\cap L}u_\alpha$ commutes with each of its terms,
(2) each u_α is self-adjoint relative to a (equivalently: $a(\alpha^\vee,z)=c_\alpha\alpha(z)$) for some $c_\alpha\in\mathbb{C}$),
(3) for every $z\in\mathfrak{h}$, b_z is self-adjoint relative to a (equivalently: $a(b(z_1,z_2),z_3)$ is symmetric in its arguments),

(4) if $\alpha(p) = 0$, then
(a) $[u_\alpha, U_p] = 0$ *and*
(b) $[u_\alpha, b_p] = 0$,

(5) if $p, q \in \Pi$ span a two dimensional face of Σ, then
(a) $[U_p, b_q] = [U_q, b_p]$ *and*
(b) $[U_p, U_q] + [b_p, b_q] = p \otimes a_q - q \otimes a_p$.

Proof. From Lemma 2.9, we can know that the connection is flat if and only if the $\tilde{\Omega}^*$-residues along the added divisors have the property that the collection of $\tilde{\Omega}^*$-residues of divisors passing through any preassigned codimension two intersection has a sum which commutes with each of its terms. We write this out for the present case.

For an intersection $E \times \mathbb{P}^1$ of two distinct divisors of type $\hat{H}_\alpha \times \mathbb{P}^1$ we get (1): the characters of H that are trivial on E make up a primitive rank two sublattice L of Q and $R \cap L$ is the set of $\alpha \in R$ for which $\hat{H}_\alpha \supset E$. Conversely, for any rank two sublattice $L \subset Q$ which contains two independent elements of R, $\cap_{\alpha \in R \cap L} \hat{H}_\alpha$ is nonempty and of codimension two in $\hat{H}$, then the sum $\sum_{\alpha \in R \cap L} u_\alpha$ commutes with each of its terms.

The intersection of $\hat{H}_\alpha \times \mathbb{P}^1$ and $D_p \times \mathbb{P}^1$ is nonempty only if $\alpha(p) = 0$ and in that case no other boundary divisor will contain that intersection; since

$$
\begin{aligned}
&[U_p + b_p + t\frac{\partial}{\partial t} \otimes a_p - p \otimes \frac{dt}{t}, u_\alpha]\\
=&[U_p + b_p, u_\alpha] + k_\alpha(a(p, \alpha^\vee)t\frac{\partial}{\partial t} \otimes \alpha\\
&- \alpha(t\frac{\partial}{\partial t})\alpha^\vee \otimes a_p - \frac{dt}{t}(\alpha^\vee)p \otimes \alpha + \alpha(p)\alpha^\vee \otimes \frac{dt}{t})\\
=&[U_p + b_p, u_\alpha] + k_\alpha a(p, \alpha^\vee)t\frac{\partial}{\partial t} \otimes \alpha,
\end{aligned}
$$

this yields $[U_p + b_p, u_\alpha] = 0$ and the condition that $a(p, \alpha^\vee) = 0$ when $\alpha(p) = 0$. Since $U_{-p} = U_p$ and $b_{-p} = -b_p$, we get (4). The hyperplane $\mathfrak{h}_\alpha$ is spanned by its intersection with Π. So the fact that $a(\alpha^\vee, p) = 0$ for all $p \in \Pi$ with $\alpha(p) = 0$ implies that $a(\alpha^\vee, y) = c_\alpha \alpha(y)$ for some c_α. This tells us that $a(u_\alpha(z), w) = c_\alpha \alpha(z)\alpha(w)$ is symmetric in z and w, in other words, u_α is self-adjoint relative to a. Conversely, if u_α is self-adjoint relative to a, then clearly, $a(\alpha^\vee, p) = 0$ when $\alpha(p) = 0$. So this amounts to (2).

The intersection of $D_p \times \mathbb{P}^1$ and $D_q \times \mathbb{P}^1$ with p and q distinct is nonempty only if p and q span a two dimensional face. In that case,

$$[U_p + b_p + t\frac{\partial}{\partial t} \otimes a_p - p \otimes \frac{dt}{t}, U_q + b_q + t\frac{\partial}{\partial t} \otimes a_q - q \otimes \frac{dt}{t}] =$$
$$[U_p + b_p, U_q + b_q] - p \otimes a_q + q \otimes a_p + t\frac{\partial}{\partial t} \otimes (a_p(U_q + b_q) - a_q(U_p + b_p)).$$

(We used that $a(p,q)$, $b(p,q)$ and $U_p(q) = -\frac{1}{2}\sum_{\alpha\in R:\alpha(p)>0,\alpha(q)>0} k_\alpha \alpha(p)\alpha(q)\alpha^\vee$ are symmetric in p and q.) We thus have $[U_p+b_p, U_q+b_q] = p\otimes a_q - q\otimes a_p$ and $(a_p(U_q + b_q) - a_q(U_p + b_p))$. If we take its invariant and anti-invariant part in the former equality,we immediately have $[U_p, U_q]+[b_p, b_q]+[U_p, b_q]-[U_q, b_p] = p \otimes a_q - q \otimes a_p$ and $[U_p, U_q] + [b_p, b_q] - [U_p, b_q] + [U_q, b_p] = p \otimes a_q - q \otimes a_p$, this yields (5). The latter is equivalent to $a(p, U_q(z)+b(q, z)) = a(q, U_p(z)+b(p, z))$ for all z. Since u_α is self-adjoint relative to a, U_p is self-adjoint relative to a as well, we then have $a(p, U_q(z)) = a(U_q(p), z) = a(U_p(q), z) = a(q, U_p(z))$. The latter condition hence simplifies to $a(p, b(q, z))$ is symmetric in p and q. Since b itself is symmetric and p and q are basis roots of $P^\vee$, we get (3).

The residue on the divisors defined by $t = 0$ and $t = \infty$ are scalars and hence yield no conditions. □

Remark 2.11. Following [**6**], Condition (1) is precisely what one needs in order that for every sublattice L of $X(H)$ spanned by elements of R the 'linearized connection' on $\mathfrak{h} - \cup_{\alpha\in R\cap L}\mathfrak{h}_\alpha$ defined by the End($\mathfrak{h}$)-valued differential

$$\Omega_L := \sum_{\alpha\in R\cap L} k_\alpha \frac{d\alpha}{\alpha} \otimes \pi_\alpha$$

be flat. According to *loc. cit.*, it is also true that the sum $\sum_{\alpha\in R\cap L} k_\alpha\pi_\alpha$ commutes with each of its terms. If a is defined over $\mathbb{R}$ and positive definite, then Conditions (1) and (2) define a Dunkl system in the sense of [**6**].

Now we need to verify these conditions of Lemma 2.10 in order to show that the connection $\tilde{\nabla}$ in our case could be flat if we choose an appropriate bilinear form a. But before we proceed to that, it is absolutely necessary to investigate the toric Lauricella case which gives a hint on these conditions.

Example 2.12 (The toric Lauricella case)**.** Let $N := \{1, 2, \cdots, n+1\}$ and assign each $i \in N$ a positive real number μ_i. Label the standard basis of $\mathbb{C}^{n+1}$ as $\varepsilon_1, \cdots, \varepsilon_{n+1}$. We endow $\mathbb{C}^{n+1}$ with a bilinear form as $a(z, w) := \sum_{i=1}^{n+1} \mu_i z^i w^i$ where z is given by $z = \sum z^i\varepsilon_i$. Let $\mathfrak{h}$ be the quotient of $\mathbb{C}^{n+1}$ by its main diagonal $\varepsilon := \mathbb{C}\sum\varepsilon_i$, but we may often identify it with the orthogonal complement of the main diagonal in $\mathbb{C}^{n+1}$, that is, with the hyperplane defined by $\sum\mu_i z^i = 0$. We take our α's to be the collection $\alpha_{i,j} := (z_i - z_j)_{i\neq j}$ where z_i is the dual basis of ε_i in $\mathfrak{h}^*$. We associate each $\alpha_{i,j}$ a $v_{i,j} := v_{z_i-z_j} := \mu_j\varepsilon_i - \mu_i\varepsilon_j$.

We immediately notice that the set $R := \{\alpha_{i,j}\}$ generates a discrete subgroup of $\mathfrak{h}^*$ whose $\mathbb{R}$-linear span defines a real form $\mathfrak{h}(\mathbb{R})$ of $\mathfrak{h}$. It's easy to show that $a(v_{i,j}, \beta) = 0$ for any $\beta \in \ker(\alpha_{i,j})$. According to [**6**], for every rank two subgroup L of the lattice generated by R, $\sum_{R\cap L} u_{\alpha_{i,j}}$ commutes with each of its terms, if $u_{\alpha_{i,j}}$ is the endomorphism of $\mathfrak{h}$ defined by $u_{\alpha_{i,j}}(z) = \alpha_{i,j}(z)v_{i,j}$.

We still denote by Π the set of primitive elements of the cocharacter lattice in spanning a face of the rational cone decomposition Σ. Then the elements of Π correspond to proper subsets I of N:

$$p_I := \frac{\mu_{I'}}{\mu_N}\varepsilon_I - \frac{\mu_I}{\mu_N}\varepsilon_{I'}$$

where I' is the complement set of I in N, $\mu_I := \sum_{i\in I}\mu_i$, and $\varepsilon_I := \sum_{i\in I}\varepsilon_i$.

Let $U_x := \sum_{\alpha_{i,j}\in R:\alpha_{i,j}(x)>0}\alpha_{i,j}(x)u_{\alpha_{i,j}}$. Notice that $\alpha_{i,j}(p_I) > 0$ if and only if $i \in I$ and $j \in I'$, its value then being 1. Put $U_I := U_{p_I}$, we thus have:

$$\begin{aligned} U_I(z) &= \sum_{i\in I, j\in I'}(z^i - z^j)(\mu_j\varepsilon_i - \mu_i\varepsilon_j) \\ &= \mu_{I'}\sum_{i\in I} z^i\varepsilon_i - \sum_{j\in I'}\mu_j z^j\sum_{i\in I}\varepsilon_i - \sum_{i\in I}\mu_i z^i\sum_{j\in I'}\varepsilon_j + \mu_i\sum_{j\in I'}z^j\varepsilon_j \\ &= (\mu_{I'}\sum_{i\in I}z^i\varepsilon_i - (\sum_{j\in I'}\mu_j z^j)\varepsilon_I) + (\mu_I\sum_{j\in I'}z^j\varepsilon_j - (\sum_{i\in I}\mu_i z^i)\varepsilon_{I'}). \end{aligned}$$

Notice that the coefficients of ε_k and ε_l are the same whenever k, l are both in I or both in I' and $z \in \alpha_{k,l}$. In other words, $\alpha_{k,l}(U_I(z)) = 0$ for $z \in \alpha_{k,l}$ whenever $\alpha_{k,l}(p_I) = 0$.

We also find that

$$U_I(p_I) = \mu_N p_I.$$

Notice that p_I and p_J span a face if and only if I and J satisfy an inclusion relation: $I \subset J$ or $I \supset J$. A straightforward computation shows that

$$\begin{aligned} U_J(p_I) = U_I(p_J) &= \mu_{J'}p_I + \mu_I p_J, \\ a(p_I, p_J) &= \frac{\mu_I\mu_{J'}}{\mu_N}, \\ [U_I, U_J](z) &= \mu_N(a(z, p_J)p_I - a(z, p_I)p_J). \end{aligned}$$

There actually exists a nonzero cubic form in this case. Let $\tilde{f} : \mathbb{C}^{n+1} \to \mathbb{C}$ be defined by $\tilde{f}(z) := \sum \mu_i(z^i)^3$ and take for $f : \mathfrak{h} \to \mathbb{C}$ its restriction to $\mathfrak{h}$. The partial derivative of $\tilde{f}$ with respect to $v_{i,j}$ is $3\mu_j\mu_i(z_i^2 - z_j^2)$, which is divisible by $\alpha_{i,j}$.

The symmetric bilinear map $\tilde{b} : \mathbb{C}^{n+1}\times\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is given by $\tilde{b}(\varepsilon_i, \varepsilon_j) := \delta_{ij}\varepsilon_i$. Then the map $b : \mathfrak{h}\times\mathfrak{h} \to \mathfrak{h}$ corresponding to f is the restriction of $\pi\circ\tilde{b}$ to $\mathfrak{h}\times\mathfrak{h}$ among which $\pi : \mathbb{C}^{n+1} \to \mathfrak{h}$ is the orthogonal projection from $\mathbb{C}^{n+1}$ to $\mathfrak{h}$. We also find that $a(\tilde{b}(z, z), z) = \tilde{f}(z)$ and $a(b(z, z), z) = f(z)$. So if we write $\tilde{b}_i(z)$ for $\tilde{b}(\varepsilon_i, z)$, we can write $\tilde{b}_i$ as $\tilde{b}_i = \varepsilon_i \otimes z_i$. If we write $b_{i,j}(z)$ for $b(v_{i,j}, z)$, then

$$b_{i,j} = \mu_j \varepsilon_i \otimes z_i - \mu_i \varepsilon_j \otimes z_j - \frac{\mu_i \mu_j}{\mu_N} \varepsilon_N \otimes (z_i - z_j).$$

If we write $a_{i,j}(z) = a(v_{i,j}, z)$, then $a_{i,j} = \mu_i \mu_j (z_i - z_j)$, we can verify that $[b_{i,j}, b_{k,l}] = -\mu_N^{-1}(v_{i,j} \otimes a_{k,l} - v_{k,l} \otimes a_{i,j})$. Hence we have $[b_z, b_w] = -\mu_N^{-1}(z \otimes a_w - w \otimes a_z)$.

We first verify the conditions about the bilinear map b in Lemma 2.10 since as Lemma 2.4 says, a nonzero b only exists for a root system of type A_n.

Lemma 2.13. *The conditions* (3), (4)(b) *and* (5)(a) *of Lemma* 2.10 *hold for a root system of type* A_n.

PROOF. Now we let all μ_i equal to 1 in the above example, then the above example becomes the case of a root system of type A_n. Since the dimension of $\mathrm{Hom}(\mathrm{Sym}^2\mathfrak{h}, \mathfrak{h})^W$ is just 1, then the $b_0 = \sum_{\alpha>0} \alpha \otimes \alpha \otimes \alpha'$ given in Lemma 2.4 is just a multiple of b given in the above example. We thus have $b_{i,j}(z) = z^i \varepsilon_i - z^j \varepsilon_j - \frac{1}{n+1}(z^i - z^j)\varepsilon_N$. If $i < j < k$, then

$$a(b_{i,j}(z), \varepsilon_j - \varepsilon_k) = -z^j = a(b_{j,k}(z), \varepsilon_i - \varepsilon_j);$$

if i, j, k, l are pairwise distinct, then

$$a(b_{i,j}(z), \varepsilon_k - \varepsilon_l) = 0.$$

Since $\{\varepsilon_i - \varepsilon_{i+1} \mid i = 1, 2, \cdots, n\}$ is a basis of $\mathfrak{h}$, Condition (3) holds.

It's obvious that u_α is self-adjoint relative to a where $v_\alpha := k_\alpha \alpha^\vee$. This is equivalent to $a(z, v_\alpha) = c_\alpha \alpha(z)$ for some $c_\alpha \in \mathbb{C}$. Since $a(b(z,z), z) = f(z)$ and $\partial_{\alpha^\vee} f$ is divisible by α for each $\alpha \in R$, there exists a $g_\alpha \in \mathfrak{h}^*$ such that $a(v_\alpha, b(z, w)) = \alpha(z) g_\alpha(w) + \alpha(w) g_\alpha(z)$. If $p \in \mathfrak{h}_\alpha$, then

$$a(w, b_p u_\alpha(z)) = \alpha(z) a(w, b(p, v_\alpha)) = \alpha(z) a(v_\alpha, b(p, w)) = \alpha(z)\alpha(w) g_\alpha(p),$$

but also

$$a(w, u_\alpha b_p(z)) = a(u_\alpha(w), b(p, z)) = \alpha(w) a(v_\alpha, b(p, z)) = \alpha(w)\alpha(z) g_\alpha(p).$$

This yields Condition (4)(b): $[u_\alpha, b_p] = 0$.

If $p \in \mathfrak{h}_\alpha(\mathbb{R})$ spans a 1-face, then $b_p u_\alpha = u_\alpha b_p$ implies that $\alpha(z) b_p(v_\alpha) = b_p u_\alpha(z) = u_\alpha b_p(z) = \alpha(b_p(z)) v_\alpha$. This shows that b_p has v_α as an eigenvector, with eigenvalue $\lambda_{p,\alpha}$, say. It could also be written as: $b_{v_\alpha}(p) = \lambda_{p,\alpha} v_\alpha$. Since $\mathfrak{h}_\alpha$ is generated by the 1-faces it contains, it follows that there is a unique linear form λ_α on $\mathfrak{h}_\alpha$ such that $b_{v_\alpha}(z) = \lambda_\alpha(z) v_\alpha$ for all $z \in \mathfrak{h}_\alpha$. Choose $v'_\alpha \in \mathfrak{h}$ such that $\lambda_\alpha(z) = a(v'_\alpha, z)$ for all $z \in \mathfrak{h}_\alpha$. We can see that this v'_α is unique up to a multiple of v_α since $\mathfrak{h}_\alpha$ is the a-orthogonal complement of v_α. So b_{v_α} has rank at most two and will be of the form $b_{v_\alpha}(z) = a(v'_\alpha, z) v_\alpha + a(v_\alpha, z) v''_\alpha$ for some

$v''_\alpha \in \mathfrak{h}$. Since b_{v_α} is self-adjoint relative to a, $a(b_{v_\alpha}(z), w) = a(v'_\alpha, z)a(v_\alpha, w) + a(v_\alpha, z)a(v''_\alpha, w)$ is symmetric in z and w. This means v''_α and v'_α differ by a multiple of v_α. So by a suitable choice of v'_α, we can arrange that $v'_\alpha = v''_\alpha$. Then

$$\begin{aligned} a(b_{v_\alpha}(z), w) &= a(v'_\alpha, z)a(v_\alpha, w) + a(v_\alpha, z)a(v'_\alpha, w) \\ &= c_\alpha(a(v'_\alpha, z)\alpha(w) + \alpha(z)a(v'_\alpha, w)). \end{aligned}$$

Let $p, q \in \mathfrak{h}(\mathbb{R})$ span a face of Σ. Then

$$\begin{aligned} &a(b_q(z), U_p(w)) \\ &\quad = -\frac{1}{2}\sum_{\alpha:\alpha(p)>0} \alpha(p)\alpha(w)a(b_q(z), v_\alpha) \\ &\quad = -\frac{1}{2}\sum_{\alpha:\alpha(p)>0} \alpha(p)\alpha(w)a(b_{v_\alpha}(z), q) \\ &\quad = -\frac{1}{2}\sum_{\alpha:\alpha(p)>0} c_\alpha\alpha(p)\alpha(w)(a(v'_\alpha, z)\alpha(q) + \alpha(z)a(v'_\alpha, q)) \\ &\quad = -\frac{1}{2}\sum_{\alpha:\alpha(p)>0} c_\alpha(\alpha(p)\alpha(q)a(v'_\alpha, z)\alpha(w) + \alpha(p)a(v'_\alpha, q)\alpha(z)\alpha(w)). \end{aligned}$$

Since U_p is self-adjoint relative to a, hence

$$\begin{aligned} a([U_p, b_q](z), w) &= a(b_q(z), U_p(w)) - a(b_q(w), U_p(z)) \\ &= -\frac{1}{2}\sum_{\alpha:\alpha(p)>0} c_\alpha\alpha(p)\alpha(q)(a(v'_\alpha, z)\alpha(w) - \alpha(z)a(v'_\alpha, w)). \end{aligned}$$

This is symmetric in p and q, because $\alpha(p) > 0$ implies $\alpha(q) \geq 0$ and the terms with $\alpha(q) = 0$ vanish. So Condition (5)(a) holds: $[U_p, b_q] = [U_q, b_p]$. □

Now we need to verify the other conditions for all the reduced irreducible root systems.

Theorem 2.14. *The connection $\tilde{\nabla}$ defined in (2.2) is flat if we choose an appropriate bilinear form a, and hence the connection ∇ defines a projective structure on H°.*

Proof. By Lemma 2.4, Conditions (3), (4)(b) and (5)(a) are empty and $[b_p, b_q] = 0$ for all types other than A_n. So we only need to verify those remaining conditions in Lemma 2.10.

Because W acts irreducibly on the space spanned by L, hence by Schur's lemma the sum $\sum_{\alpha\in R\cap L} u_\alpha$ acts as a scalar operator, Condition (1) is thus satisfied. And there always exists a nondegenerate symmetric bilinear form

$a : \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}$ such that $\alpha^\vee$ is a-perpendicular to $\ker(\alpha)$, i.e., $a(\alpha^\vee, p) = 0$ if $\alpha(p) = 0$. This implies Condition (2).

As the reflection s_α and u_α have the relation: $u_\alpha = k_\alpha(1 - s_\alpha)$, u_α commutes with U_p is equivalent to that s_α commutes with U_p. While $s_\alpha u_\beta s_\alpha^{-1} = u_{s_\alpha(\beta)}$, we have

$$\begin{aligned}
s_\alpha U_p s_\alpha^{-1} &= s_\alpha(-\frac{1}{4}\sum_{\beta\in R}|\beta(p)|u_\beta)s_\alpha^{-1} \\
&= -\frac{1}{4}\sum_{\beta\in R}|s_\alpha^2(\beta)(p)|u_{s_\alpha(\beta)} \\
&= -\frac{1}{4}\sum_{\beta'\in R}|s_\alpha(\beta')(p)|u_{\beta'} \qquad \text{(here we let } \beta' = s_\alpha(\beta)) \\
&= -\frac{1}{4}\sum_{\beta'\in R}|\beta'(s_\alpha(p))|u_{\beta'} \\
&= -\frac{1}{4}\sum_{\beta'\in R}|\beta'(p)|u_{\beta'} \qquad \text{(here } s_\alpha(p) = p \text{ because } \alpha(p) = 0) \\
&= U_p.
\end{aligned}$$

So Condition (3) follows.

If $p \in \mathfrak{h}_\alpha$, then U_p preserves $\mathfrak{h}_\alpha$ and since U_p is self-adjoint relative to a, U_p will have $\alpha^\vee$ as an eigenvector. Since $\mathbb{C}p + \mathbb{C}q$ is an intersection of hyperplanes $\mathfrak{h}_\alpha$, the a-orthogonal complement of $\mathbb{C}p + \mathbb{C}q$ is spanned by the vectors $\alpha^\vee$ it contains. Hence we have a common eigenspace decomposition of this subspace for U_p and U_q. In particular, these endomorphisms commute there. Since $[U_p, U_q]$ is an element of the Lie algebra of the orthogonal group of a whose kernel contains the a-orthogonal complement of $\mathbb{C}p + \mathbb{C}q$, it is necessarily a multiple of $p \otimes a_q - q \otimes a_p$, i.e., $[U_p, U_q] = \lambda(p \otimes a_q - q \otimes a_p)$. But for each pair of (p, q), we can choose a chamber C, a member of Σ that is open and nonempty in $\mathfrak{h}(\mathbb{R})$, and this pair of (p, q) could be pulled back by an element of W to some 2-face of the closure of the chamber $\bar{C}$. Notice U_x is continuous on $\bar{C}$ and the map $(x, y) \mapsto [U_x, U_y]$ is bilinear on $\bar{C} \times \bar{C}$, we can know that all the pair of $[U_p, U_q]$ share the same coefficient of λ. In particular, for R of types other than A_n, we can normalize a such that λ becomes equal to 1. For R of type A_n, we know that $[b_p, b_q]$ is also a multiple of $p \otimes a_q - q \otimes a_p$ and share the same coefficient μ for any pair of (p, q) from Example 2.12, so we can also normalize a such that $\lambda + \mu = 1$. Then, Condition (4) is satisfied. □

In fact, we can write out the explicit form of a^κ in terms of a given inner product $(\cdot, \cdot)$ according to the Condition (5)(b) of Lemma 2.10 if we want to construct a projective structure on H°.

Theorem 2.15. *If we use the construction of root systems in Bourbaki and take the inner product $(\cdot,\cdot)$ such that $(\varepsilon_i,\varepsilon_j)=\delta_{ij}$, then the a^κ such that $\tilde{\nabla}^\kappa$ is flat is given as follows:*

$$\begin{aligned}
&A_n : a^\kappa(u,v) = \frac{(n+1)}{4}(k^2-k'^2)(u,v);\\
&B_n : a^\kappa(u,v) = ((n-2)k^2+kk')(u,v);\\
&C_n : a^\kappa(u,v) = ((n-2)k^2+2kk')(u,v);\\
&D_n : a^\kappa(u,v) = (n-2)k^2(u,v);\\
&E_n : a^\kappa(u,v) = ck^2(u,v); \quad c=6,12,30 \mathit{\ for\ } n=6,7,8;\\
&F_4 : a^\kappa(u,v) = (k+k')(2k+k')(u,v);\\
&G_2 : a^\kappa(u,v) = \frac{3}{4}(k+3k')(k+k')(u,v).
\end{aligned}$$

Proof. We already know that a^κ is a multiple of the given inner product by the Schur's lemma if R is irreducible. Then it's a straightforward computation by the Condition (5)(b) of Lemma 2.10: $[U_p,U_q]+[b_p,b_q]=p\otimes a^\kappa_q - q\otimes a^\kappa_p$. In fact, the a^κ for type A_n can be obtained directly from Example 2.12.

Let's determine the a^κ for type C_n for example. Put $p:=\varepsilon_1+\cdots+\varepsilon_s$ and $q:=\varepsilon_1+\cdots+\varepsilon_t$. Assume that $s<t$ without loss of generality. It's obvious that $(p,p)=s$ and $(p,q)=s$. A straightforward computation shows that

$$\begin{aligned}
&U_p(\varepsilon_m) = -(((n-2)k+2k')\varepsilon_m+kp) \quad \text{for } 1\le m\le s;\\
&U_p(\varepsilon_m) = -sk\varepsilon_m \quad \text{for } s+1\le m\le n.
\end{aligned}$$

Then we have

$$\begin{aligned}
&U_p(p) = -((n-2+s)k+2k')p;\\
&U_p(q) = U_q(p) = -((n-2)k+2k')p+2skq);\\
&U_q(q) = -((n-2+t)k+2k')q.
\end{aligned}$$

Hence

$$[U_p,U_q](p) = sk((n-2)k+2k')p - sk((n-2)k+2k')q.$$

We thus have

$$a^\kappa(u,v) = ((n-2)k^2+2kk')(u,v).$$

The remaining cases are left to the readers. $\square$

Therefore, we have constructed a W-invariant projective structure on H° where H is an adjoint torus.

CHAPTER 3

Hyperbolic structures

In this chapter, we show that the toric arrangement complement H° admits a hyperbolic structure when κ lies in some region so that its image under the projective evaluation map lands in a complex ball. In Section 3.1, we review the basic theory of geometric structures with logarithmic singularities. In Section 3.2, we compute the eigenvalues of the residue endomorphisms along those added divisors, which almost equals to obtaining the logarithmic exponents along those divisors. In Section 3.3, we use the method of reflection representation to investigate the corresponding Hermitian form so that we can determine the hyperbolic region for H°. In Section 3.4, we set up the (projective) evaluation map and even give out the evaluation map around those divisors in the form of local coordinates. In Section 3.5, we provide a proof showing that the dual Hermitian form is greater than 0 when κ lies in the hyperbolic region which means its image under the projective evaluation map lands in a complex ball. In the final section, we show an example of ball quotients, which actually strongly motivated current research.

3.1. Geometric structures with logarithmic singularities

We shall in this section introduce the geometric structures on a complex manifold in a very brief way. A good exposition on this topic is Chapter 1 of [**6**].

Let $\tilde{M} \to M$ be a holonomy covering and denote by Γ its Galois group. So $\mathrm{Aff}(\tilde{M}) := H^0(\tilde{M}, \mathrm{Aff}_{\tilde{M}})$ is a Γ-invariant vector space of holomorphic functions on $\tilde{M}$. Then the set A of linear forms $\mathrm{Aff}(\tilde{M}) \to \mathbb{C}$ which are the identity on $\mathbb{C}$ is an affine Γ-invariant hyperplane in $\mathrm{Aff}(\tilde{M})^*$.

Definition 3.1. Given a holonomy cover as above, the evaluation map ev : $\tilde{M} \to A$ which assigns to $\tilde{z}$ the linear form $ev_{\tilde{z}} \in A : \mathrm{Aff}(\tilde{M}) \to \mathbb{C}; \tilde{f} \mapsto \tilde{f}(\tilde{z})$ is called the *developing map* of the affine structure; it is Γ-equivariant and a local affine isomorphism.

This tells us that a developing map determines a natural affine atlas on M whose charts take values in A and whose transition maps lie in Γ.

Definition 3.2. Suppose an affine structure is given on a complex manifold M by a torsion free, flat connection ∇. We call a nowhere zero holomorphic vector field E on M a *dilatation field* with factor $\lambda \in \mathbb{C}$ such that $\nabla_X(E) = \lambda X$ for every local vector field X.

If X is flat, then the torsion freeness yields: $[E, X] = \nabla_E(X) - \nabla_X(E) = -\lambda X$. This tells us that Lie derivative with respect to E acts on flat vector fields simply as multiplication by $-\lambda$. Hence it acts on flat differentials as multiplication by λ.

Let h be a flat Hermitian form on the tangent bundle of M such that $h(E, E)$ is nowhere zero. Then the leaf space M/E of the dimension one foliation defined by E inherits a Hermitian form $h_{M/E}$ in much the same way as the projective space of a finite dimensional Hilbert space acquires its Fubini-Study metric. We are especially interested in the case when $h_{M/E}$ is positive definite:

Definition 3.3. Let M be a complex manifold with an affine structure and there is a dilatation field E on M with factor λ. We say that a flat Hermitian form h on M is *admissible* relative to E if it is in one of the following three cases:

(1) elliptic: $\lambda \neq 0$ and $h > 0$;
(2) parabolic: $\lambda = 0$ and $h \geq 0$ with kernel spanned by E;
(3) hyperbolic: $\lambda \neq 0$, $h(E, E) < 0$ and $h > 0$ on $E^{\perp}$.

Then the leaf space M/E acquires a metric $h_{M/E}$ of constant holomorphic sectional curvature, for it is locally isometric to a complex projective space with Fubini-Study metric, to a complex-Euclidean space or to a complex-hyperbolic space respectively.

In order to understand the behavior of an affine structure near a given smooth subvariety of its singular locus, we need to blow up that subvariety so that we are dealing with the codimension one case. Let's first look at the simplest degenerating affine structures as follows which is also in [**6**].

Definition 3.4. Let D be a smooth connected hypersurface in a complex manifold M and let be given an affine structure on $M - D$. We say that the affine structure on $M - D$ has an *infinitesimally simple degeneration along* D of logarithmic exponent $\lambda \in \mathbb{C}$ if

(1) ∇ extends to $\Omega_M(\log D)$ with a logarithmic pole along D,
(2) the residue of this extension along D preserves the subsheaf $\Omega_D \subset \Omega_M(\log D) \otimes \mathcal{O}_D$ and its eigenvalue on the quotient sheaf $\mathcal{O}_D$ is λ and
(3) the residue endomorphism restricted to Ω_D is semisimple and all of its eigenvalues are λ or 0.

We have the following local model for the behavior of the developing map for such a degenerating affine structure [**6**].

Proposition 3.5. *Let be given a smooth hypersurface D in a complex manifold M, an affine structure on $M - D$ and $p \in D$. Then the affine structure has an infinitesimally simple degenerating along D at p of logarithmic exponent $\lambda \in \mathbb{C}$ if and only if there exists a local equation t for D and a local chart as follows*

$$(F_0, t, F_\lambda) : M_p \to (T_0 \times \mathbb{C} \times T_\lambda)_{(0,0,0)}$$

(T_λ incorporates into T_0 when $\lambda = 0$), where T_0 and T_λ are vector spaces, such that the developing map near p is affine equivalent to the following multivalued map with range $T_0 \times \mathbb{C} \times T_\lambda$):

$$\begin{aligned}
\lambda \notin \mathbb{Z} \quad &: \quad (F_0, t^{-\lambda}, t^{-\lambda}F_\lambda), \\
\lambda \in \mathbb{Z}_+ \quad &: \quad (F_0, t^{-\lambda}, t^{-\lambda}F_\lambda) + \log t.(0, A \circ F_0), \\
&\quad\ \ \textit{where } A : T_0 \to \mathbb{C} \times T_\lambda \textit{ is an affine-linear map}, \\
\lambda \in \mathbb{Z}_- \quad &: \quad (F_0, t^{-\lambda}, t^{-\lambda}F_\lambda) + \log t.t^{-\lambda}(B \circ F_\lambda, 0, 0), \\
&\quad\ \ \textit{where } B : T_\lambda \to T_0 \textit{ is an affine-linear map}, \\
\lambda = 0 \quad &: \quad (F_0, \log t.\alpha \circ F_0), \\
&\quad\ \ \textit{where } \alpha : T_0 \to \mathbb{C} \textit{ with } \alpha(0) \neq 0 \textit{ is an affine-linear function.}
\end{aligned}$$

When $\lambda \notin \mathbb{Z}$, the holonomy around D_p (and hence the monodromy around D_p) is semisimple. When $\lambda \in \mathbb{Z} - \{0\}$, the monodromy is semisimple if and only if the associated affine-linear map is zero (and in that case the holonomy is equal to the identity). When $\lambda = 0$, the monodromy is semisimple if and only if α is constant and in that case the holonomy is a translation.

In fact, we need to understand what happens in case D is a normal crossing divisor in the complex manifold M and the affine structure on $M - D$ degenerates infinitesimally simply along some irreducible component of D

Proposition 3.6. ([**6**]) *Let be given a simple normal crossing divisor D in a complex manifold M with smooth irreducible components $D_1, \cdots, D_k$ and an affine structure on $M - D$ with infinitesimally simply degeneration along D_i of logarithmic exponent λ_i. Suppose that no λ_i is a negative integer, that the holonomy around D_i is semisimple unless $\lambda_i = 0$ and that for any pair $1 \leq i < j \leq k$, the formation of the local affine quotient of the generic point of D_i extends across the generic point of $D_i \cap D_j$. Let $p \in \cap D_i$. Then $\lambda_i \neq 0$ for $i < k$ and the local affine retraction r_i at the generic point of D_i extends to $r_i : M_p \to D_{i,0}$ in such a manner that $r_i r_j = r_i$ for $i < j$. Furthermore, there exists a local equation t_i for D_i and a morphism to a vector space $F_i : M_p \to T_i$ with $F_i(p) = 0$ such that these are the components of a chart for M_p and are* *such that the developing map is affine equivalent to the multivalued map*

$$(F_0, (t_1^{-\lambda_1}\cdots t_i^{-\lambda_i}(1,F_i))_{i=1}^k) \,:\, M_p \to T_0 \times \prod_{i=1}^k (\mathbb{C}\times T_i) \quad \textit{when} \quad \lambda_k \neq 0;$$

$$(F_0, (t_1^{-\lambda_1}\cdots t_i^{-\lambda_i}(1,F_i))_{i=1}^{k-1}, t_1^{-\lambda_1}\cdots t_{k-1}^{-\lambda_{k-1}}\log t_k) :$$
$$M_p \to T_0 \times \prod_{i=1}^{k-1}(\mathbb{C}\times T_i)\times\mathbb{C} \quad \textit{when} \quad \lambda_k = 0.$$

3.2. Eigenvalues of the residue endomorphisms

We can see from the preceding section that it is important to know that the eigenvalues of the residue maps of the connection $\tilde{\nabla}^\kappa$. Those residues are as follows from the last chapter.

$$\begin{aligned}
\mathrm{Res}_{\hat{H}_\alpha\times\mathbb{P}^1}(\tilde{\Omega}^\kappa)^* &= u_\alpha,\\
\mathrm{Res}_{D_p\times\mathbb{P}^1}(\tilde{\Omega}^\kappa)^* &= U_p + b_p^\kappa + t\frac{\partial}{\partial t}\otimes a_p^\kappa - p\otimes\frac{dt}{t},\\
\mathrm{Res}_{t=0}(\tilde{\Omega}^\kappa)^* &= -\mathrm{Res}_{t=\infty}(\tilde{\Omega}^\kappa)^* = -1_{\mathfrak{h}\oplus\mathbb{C}}.
\end{aligned}$$

Now let us compute the eigenvalues of these residues. We look at the residue map along the mirror $\hat{H}_\alpha\times\mathbb{P}^1$, regard u_α as an endomorphism of $\mathfrak{h}$ instead of $\mathfrak{h}\oplus\mathbb{C}$ at first. We immediately have

$$\begin{aligned}
u_\alpha(\alpha^\vee) &= 2k_\alpha\alpha^\vee,\\
u_\alpha(p) &= 0 \quad \text{for} \quad \forall p\perp\alpha^\vee.
\end{aligned}$$

Then we have the following eigenvalues

$$\begin{cases}
u_\alpha((\alpha^\vee,0)) = 2k_\alpha(\alpha^\vee,0),\\
u_\alpha((p,0)) = 0 \quad \text{for} \quad \forall p\perp\alpha^\vee,\\
u_\alpha((0,\lambda t\frac{\partial}{\partial t})) = 0
\end{cases}$$

if we regard u_α as an endomorphism of $\mathfrak{h}\oplus\mathbb{C}$.

Then we need to compute the eigenvalues of the residue maps of the connection along the boundary divisor $D_p\times\mathbb{P}^1$, which is much more involved. Let us first look at an example. Then we shall have some feeling about how these eigenvalues come up.

Example 3.7. We take type A_n and regard U_p and b_p as endomorphisms of $\mathfrak{h}$ at first as before. Here we still use the construction for root systems from Bourbaki. Let root system R of type A_n sit inside a Euclidean space $\mathbb{R}^{n+1}$

and denote its orthonormal basis by $e^1, e^2, \cdots, e^{n+1}$, so its positive roots are all of the form $e^i - e^j$ for $1 \leq i < j \leq n+1$. Its dual root system $R^\vee$ is also of type A_n and we denote the dual orthonormal basis by $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n+1}$. Then its positive coroots are all of the form $\varepsilon_i - \varepsilon_j$ for $1 \leq i < j \leq n+1$ and simple coroots are of the form of $\varepsilon_i - \varepsilon_{i+1}$ for $i = 1, 2, \cdots, n$.

Let $p = \varpi_m^\vee := \frac{n+1-m}{n+1}(\varepsilon_1 + \cdots + \varepsilon_m) - \frac{m}{n+1}(\varepsilon_{m+1} + \cdots + \varepsilon_{n+1})$. We know that all the positive roots α of R such that $\alpha(p) \neq 0$ are of the form $e^i - e^j$ for $1 \leq i \leq m$, $m+1 \leq j \leq n+1$ and in fact $\alpha(p) = 1$ for all these α's. Write

$$\begin{aligned} \sigma_0 &= \sum_{\alpha \in R_+} |\alpha(p)|(\alpha^\vee \otimes \alpha) \\ &= \sum_{\substack{1 \leq i \leq m \\ m+1 \leq j \leq n+1}} (\varepsilon_i - \varepsilon_j) \otimes (e^i - e^j), \end{aligned}$$

we then have

$$\sigma_0(\varepsilon_s) = \begin{cases} \displaystyle\sum_{m+1 \leq j \leq n+1} (\varepsilon_s - \varepsilon_j) & \text{for} \quad 1 \leq s \leq m \\ \displaystyle\sum_{1 \leq i \leq m} -(\varepsilon_i - \varepsilon_s) & \text{for} \quad m+1 \leq s \leq n+1. \end{cases}$$

After a straightforward computation, we have

$$\begin{cases} \sigma_0(\varepsilon_s - \varepsilon_t) = (n+1-m)(\varepsilon_s - \varepsilon_t) & \text{for} \quad 1 \leq s < t \leq m \\ \sigma_0(p) = (n+1)p \\ \sigma_0(\varepsilon_s - \varepsilon_t) = m(\varepsilon_s - \varepsilon_t) & \text{for} \quad m+1 \leq s < t \leq n+1. \end{cases}$$

Since $U_p = -\frac{1}{4}\sum_{\alpha \in R} |\alpha(p)| k(\alpha^\vee \otimes \alpha) = -\frac{1}{2}k\sigma_0$, then the above tells us that

$$\begin{cases} U_p(\alpha_i^\vee) = -\dfrac{1}{2}(n+1-m)k\alpha_i^\vee & \text{for} \quad 1 \leq i \leq m-1 \\ U_p(p) = -\dfrac{1}{2}(n+1)kp \\ U_p(\alpha_i^\vee) = -\dfrac{1}{2}mk\alpha_i^\vee & \text{for} \quad m+1 \leq i \leq n. \end{cases}$$

Since $b_p = \frac{1}{2}k' \sum_{\alpha > 0} \alpha(p)(\alpha' \otimes \alpha)$ where $\alpha' = \varepsilon_i + \varepsilon_j - \frac{2}{n+1}(\varepsilon_1 + \cdots + \varepsilon_{n+1})$, the computation for b_p is similar to the situation of U_p and hence we have

$$\begin{cases} b_p(\alpha_i^\vee) = \dfrac{1}{2}(n+1-m)k'\alpha_i^\vee & \text{for} \quad 1 \leq i \leq m-1 \\ b_p(p) = \dfrac{1}{2}(n+1-2m)k'p \\ b_p(\alpha_i^\vee) = -\dfrac{1}{2}mk'\alpha_i^\vee & \text{for} \quad m+1 \leq i \leq n. \end{cases}$$

If we write $\sigma = \mathrm{Res}_{D_p\times\mathbb{P}^1}(\tilde{\Omega}^\kappa)^* = U_p + b_p + t\frac{\partial}{\partial t}\otimes a_p - p\otimes\frac{dt}{t}$, regard U_p and b_p as endomorphisms of $\mathfrak{h}\oplus\mathbb{C}$, we then have the following eigenvalues after a little bit more effort

$$\begin{cases} \sigma((\alpha_i^\vee,0)) = -\frac{1}{2}(n+1-m)(k-k')(\alpha_i^\vee,0) \quad \text{for} \quad 1\le i\le m-1 \\ \sigma((p,-\frac{1}{2}m(k+k')t\frac{\partial}{\partial t})) = -\frac{1}{2}(n+1-m)(k-k')(p,-\frac{1}{2}m(k+k')t\frac{\partial}{\partial t}) \\ \sigma((p,-\frac{1}{2}(n+1-m)(k-k')t\frac{\partial}{\partial t})) = -\frac{1}{2}m(k+k')(p,-\frac{1}{2}(n+1-m)(k-k')t\frac{\partial}{\partial t}) \\ \sigma((\alpha_i^\vee,0)) = -\frac{1}{2}m(k+k')(\alpha_i^\vee,0) \quad \text{for} \quad m+1\le i\le n. \end{cases}$$

From this example, we notice that the eigenvalue of $U_p(+b_p)$ on the space $\mathbb{C}p$ is the sum of the two eigenvalues on the spaces $\mathfrak{h}_1 = \mathrm{span}\{\alpha_1^\vee,\cdots,\alpha_{m-1}^\vee\}$ and $\mathfrak{h}_2 = \mathrm{span}\{\alpha_{m+1}^\vee,\cdots,\alpha_n^\vee\}$ respectively. And the product of the two eigenvalues is $a(p,p)$. In fact, this holds for all the root systems. In the end, σ has two eigenvalues on the space $\mathfrak{h}\oplus\mathbb{C}$.

Theorem 3.8. *Let $\sigma = \mathrm{Res}_{D_p\times\mathbb{P}^1}(\tilde{\Omega}^\kappa)^* = U_p + b_p + t\frac{\partial}{\partial t}\otimes a_p - p\otimes\frac{dt}{t}$, then σ has at most two eigenvalues on the space $\mathfrak{h}\oplus\mathbb{C}$ with multiplicites m and $n+1-m$ respectively. In fact, these two eigenvalues satisfy such a quadratic equation $\lambda^2-\varphi\lambda+a(p,p)=0$ where φ is the eigenvalue of $U_p(+b_p)$ (if regarded as an endomorphism of $\mathfrak{h}$) on $\mathbb{C}p$.*

Proof. For type A_n, from above example, the Theorem holds.

For other types, $b_p=0$, suppose $p=\varpi_m^\vee$, we have a decomposition of $\mathfrak{h}$:

$$\mathfrak{h} = \mathbb{C}p\oplus\sum_i \mathfrak{h}_i$$

where $\mathfrak{h}_i$ is just the space spanned by the irreducible root subsystem after deleting the m-th node from the original root system R. These subspaces have the corresponding Weyl groups, denoted by W_i respectively. In fact, U_p is a W_i-invariant endomorphism in $\mathfrak{h}_i$, so U_p is just a scalar action in $\mathfrak{h}_i$ by Schur's lemma. We write $U_p(v)=\lambda_i v$ if $v\in\mathfrak{h}_i$ and $U_p(p)=\varphi_1 p$.

We check the square of σ, we have

$$\sigma^2 = U_p^2 - p\otimes a_p - U_p(p)\otimes\frac{dt}{t} + t\frac{\partial}{\partial t}\otimes a_p(U_p) - a(p,p)t\frac{\partial}{\partial t}\otimes\frac{dt}{t}.$$

From the computation above and below, we find that for all the root systems, we have (at most) two eigenvalues on the space perpendicular to p whose sum is φ_1 and product $a(p,p)$, i.e.,

$$U_p^2(q) - \varphi_1 U_p(q) + a(p,p)q = 0 \quad \text{for} \quad \forall q\in p^\perp$$

Then we can easily check that

$$\sigma^2 - \varphi_1 \sigma + a(p, p) = 0.$$

The multiplicities follow from the decomposition of the root system. $\square$

Remark 3.9. The value m and $n+1-m$ for the multiplicities in the above theorem is true except for an extremal node of D_n and some nodes of E_n, which you can see from below. In fact, we only need to bear in mind that the multiplicities follow from the decomposition of the root system in nature.

The computation for all the other types are similar, we just list the results over here.

For type B_n, corresponding to $p = \varepsilon_1 + \cdots + \varepsilon_m$ for $1 \le m \le n$, we have

$$\begin{cases} \sigma((\alpha_i^\vee, 0)) = -((n-2)k + k')(\alpha_i^\vee, 0) \quad \text{for} \quad 1 \le i \le m-1 \\ \sigma((p, -mkt\frac{\partial}{\partial t})) = -((n-2)k + k')(p, -mkt\frac{\partial}{\partial t}) \\ \sigma((p, -((n-2)k + k')t\frac{\partial}{\partial t})) = -mk(p, -((n-2)k + k')t\frac{\partial}{\partial t}) \\ \sigma((\alpha_i^\vee, 0)) = -mk(\alpha_i^\vee, 0) \quad \text{for} \quad m+1 \le i \le n. \end{cases}$$

For type C_n, corresponding to $p = \varepsilon_1 + \cdots + \varepsilon_m$ for $1 \le m < n$, we have

$$\begin{cases} \sigma((\alpha_i^\vee, 0)) = -((n-2)k + 2k')(\alpha_i^\vee, 0) \quad \text{for} \quad 1 \le i \le m-1 \\ \sigma((p, -mkt\frac{\partial}{\partial t})) = -((n-2)k + 2k')(p, -mkt\frac{\partial}{\partial t}) \\ \sigma((p, -((n-2)k + 2k')t\frac{\partial}{\partial t})) = -mk(p, -((n-2)k + 2k')t\frac{\partial}{\partial t}) \\ \sigma((\alpha_i^\vee, 0)) = -mk(\alpha_i^\vee, 0) \quad \text{for} \quad m+1 \le i \le n; \end{cases}$$

and corresponding to $p = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n)$, we have

$$\begin{cases} \sigma((\alpha_i^\vee, 0)) = -\frac{1}{2}((n-2)k + 2k')(\alpha_i^\vee, 0) \quad \text{for} \quad 1 \le i \le n-1 \\ \sigma((p, -\frac{1}{2}nkt\frac{\partial}{\partial t})) = -\frac{1}{2}((n-2)k + 2k')(p, -\frac{1}{2}nkt\frac{\partial}{\partial t}) \\ \sigma((p, -\frac{1}{2}((n-2)k + 2k')t\frac{\partial}{\partial t})) = -\frac{1}{2}nk(p, -\frac{1}{2}((n-2)k + 2k')t\frac{\partial}{\partial t}). \end{cases}$$

For type D_n, corresponding to $p = \varepsilon_1 + \cdots + \varepsilon_m$ for $1 \le m \le n-2$, we have

$$\begin{cases} \sigma((\alpha_i^\vee, 0)) = -(n-2)k(\alpha_i^\vee, 0) \quad \text{for} \quad 1 \le i \le m-1 \\ \sigma((p, -mkt\dfrac{\partial}{\partial t})) = -(n-2)k(p, -mkt\dfrac{\partial}{\partial t}) \\ \sigma((p, -(n-2)kt\dfrac{\partial}{\partial t})) = -mk(p, -(n-2)kt\dfrac{\partial}{\partial t}) \\ \sigma((\alpha_i^\vee, 0)) = -mk(\alpha_i^\vee, 0) \quad \text{for} \quad m+1 \le i \le n; \end{cases}$$

and corresponding to $p = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n)$ or $p = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{n-1} + \varepsilon_n)$, we have

$$\begin{cases} \sigma((\alpha_i^\vee, 0)) = -\dfrac{1}{2}(n-2)k(\alpha_i^\vee, 0) \quad \text{for} \quad \forall \alpha_i^\vee \perp p \\ \sigma((p, -\dfrac{1}{2}nkt\dfrac{\partial}{\partial t})) = -\dfrac{1}{2}(n-2)k(p, -\dfrac{1}{2}nkt\dfrac{\partial}{\partial t}) \\ \sigma((p, -\dfrac{1}{2}(n-2)kt\dfrac{\partial}{\partial t})) = -\dfrac{1}{2}nk(p, -\dfrac{1}{2}(n-2)kt\dfrac{\partial}{\partial t}). \end{cases}$$

For type F_4, the eigenvalues are $\{-(m+1)(k+k'), -m(2k+k')\}$ corresponding to $p = \varpi_m^\vee$ for $m = 1, 2, 3$ and $\{-2(k+k'), -2(2k+k')\}$ for $p = \varpi_4^\vee$.

For type G_2, the eigenvalues are $\{-(k+3k'), -\frac{3}{2}(k+k')\}$ and $\{-\frac{1}{2}(k+3k'), -(k+k')\}$ for $p = \varpi_1^\vee$ and $\varpi_2^\vee$ respectively.

For type E_n, the computation becomes more complicated since the construction for their fundamental coweights is somehow irregular one by one. Then in order to compute their eigenvalues, we have the following observation: the collection of $\alpha \in R$ with $\alpha(p) > 0$ is a union of W_p-orbits. So if we put for any W_p-orbit of roots that are positive on p as follows:

$$E_O := \frac{1}{2|W_p|} \sum_{w \in W_p} w\alpha^\vee \otimes w\alpha = \frac{1}{2|O|} \sum_{\alpha' \in O} \alpha'^\vee \otimes \alpha'$$

where α is a member of O, then U_p is a linear combination of such E_O's.

We denote the orthogonal complement of p by $\bar{\mathfrak{h}}$. Since E_O is a W_p-invariant endomorphism in each summand $\mathfrak{h}_i$, E_O is a scalar action on $\mathbb{C}p$ and each summand $\mathfrak{h}_i$ by Schur's lemma. These eigenvalues only depend on the W_p-orbit O of α and so we denote them by $\lambda_{p,O}$ and $\lambda_{i,O}$. First we notice that the trace of E_O is equal to $\frac{1}{2}\alpha(\alpha^\vee) = 1$ and we shall show that the traces of E_O on these eigenspaces are distributed in a simple manner. First we observe that

$$a(E_O(x), y) = \frac{a(\alpha^\vee, \alpha^\vee)}{4|W_p|} \sum_{w \in W_p} \alpha(wx) \cdot \alpha(wy).$$

When $x = y = p$ the left hand side is $a(p,p)\lambda_{p,O}$ and the right hand side becomes $\frac{1}{4}a(\alpha^\vee,\alpha^\vee)\alpha(p)^2 = a(\alpha^\vee,p)^2/a(\alpha^\vee,\alpha^\vee)$ and so

$$\lambda_{p,O} = \frac{a(\alpha,p)^2}{a(p,p)a(\alpha^\vee,\alpha^\vee)} = \frac{\parallel \pi_p(\alpha^\vee) \parallel_a^2}{\parallel \alpha^\vee \parallel_a^2},$$

where $\parallel \ \parallel_a$ denotes the norm associated to a. This is just the cosine squared of the angle between $\alpha^\vee$ and $\pi_p(\alpha^\vee)$.

When $x = y \in \mathfrak{h}_i$, the left hand side is $a(x,x)\lambda_{i,O}$. Now we only consider the case for which R is a single W-orbit for simplicity. Then we have $R_i := R \cap \mathfrak{h}_i^*$ is a W_p-orbit. We denote the Coxeter number of $W(R_i)$ by h_i. It is known that $|R_i| = h_i \dim \mathfrak{h}_i$ and that $\sum_{\beta \in R_i} \beta \otimes \beta$ is a $W(R_i)$-invariant form on $\mathfrak{h}_i$ which gives each coroot the squared length $4h_i$. So if we take in the above formula $x = y$ a coroot of R_i, then

$$\begin{aligned}
\lambda_{i,O} &= \frac{1}{4|W_p|}\sum_{w \in W_p} \alpha(w\beta^\vee)^2 = \frac{1}{4|R_i|}\sum_{\beta \in R_i} \alpha(\beta^\vee)^2 \\
&= \frac{1}{4|R_i|}\sum_{\beta \in R_i} \beta(\alpha^\vee)^2 = \frac{1}{4|R_i|} \cdot 4h_i \frac{\parallel \pi_{\mathfrak{h}_i}(\alpha^\vee) \parallel_a^2}{\parallel \text{coroot} \parallel_a^2} = \frac{\parallel \pi_{\mathfrak{h}_i}(\alpha^\vee) \parallel_a^2}{\dim \mathfrak{h}_i \parallel \text{coroot} \parallel_a^2}.
\end{aligned}$$

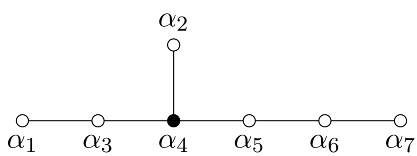
We can see that the trace of E_O on $\mathfrak{h}_i$ ($= \lambda_{i,O} \dim \mathfrak{h}_i$) is the cosine squared of the angle between $\alpha^\vee$ and $\pi_{\mathfrak{h}_i}(\alpha^\vee)$.

Then we look at these orbits. Let $\tilde{\alpha}$ be the highest root of R relative to the root basis $\mathfrak{B}$ and put $n_p := \tilde{\alpha}(p)$. By inspection one finds that for $c = 1, 2, \cdots, n_p$ the set of $\alpha \in R$ with $\alpha(p) = c$ make up a single W_p-orbit $O(c)$ and that the orthogonal projection $O(c)_i$ of $O(c)$ in $\mathfrak{h}_i^*$ is either $\{0\}$ or the orbit of a fundamental weight of R_i. So the W_p-orbit $O(c)$ projects in $\bar{\mathfrak{h}}$ bijectively onto $\prod_i O(c)_i$. We also see that there is a unique $\alpha(c) \in O(c)$ such that $\alpha(c)$ defines a fundamental coweight in $\mathfrak{h}_i$ relative to $\mathfrak{B}_i := \mathfrak{B} \cap R_i$ or 0. Therefore, our U_p is proportional to

$$E := \sum_{c=1}^{n_p} c|O(c)|E_{O(c)}.$$

Example 3.10. We do a branch point of type E_7. Let p be chosen corresponding to α_4:

E_7



The decomposition of $\mathfrak{B} - \alpha_4$ into irreducible root basis is $\mathfrak{B}_1 := \{\alpha_1, \alpha_3\}$, $\mathfrak{B}_2 := \{\alpha_2\}$, and $\mathfrak{B}_3 := \{\alpha_5, \alpha_6, \alpha_7\}$ (of type A_2, A_1 and A_3 respectively).

The highest root is $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$. Since $\tilde{\alpha}(p) = 4$, we have 4 corresponding W_p-orbits and each of them is represented by

$$\begin{aligned}
\alpha(1) &:= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \\
\alpha(2) &:= \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, \\
\alpha(3) &:= \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, \\
\alpha(4) &:= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7.
\end{aligned}$$

For the case $c = 1$, observe that $\alpha(1)$ defines the coweight sum $p_1(\mathfrak{B}_1) + p_2(\mathfrak{B}_2) + p_7(\mathfrak{B}_3)$ in $\bar{\mathfrak{h}}$. We have

$$\frac{\| \pi_p(\alpha(1)) \|^2}{\| \alpha(1) \|^2} = \frac{1}{24}, \quad \frac{\| p_1(\mathfrak{B}_1) \|^2}{\| \alpha(1) \|^2} = \frac{1}{3}, \quad \frac{\| p_2(\mathfrak{B}_2) \|^2}{\| \alpha(1) \|^2} = \frac{1}{4}, \quad \frac{\| p_7(\mathfrak{B}_3) \|^2}{\| \alpha(1) \|^2} = \frac{3}{8}$$

(summing to 1). So the eigenvalues of $E_{O(1)}$ are $(\frac{1}{24}, \frac{1}{6}, \frac{1}{4}, \frac{1}{8})$. It's easy to see that $|O(1)| = 2 \cdot 3 \cdot 4 = 24$ and so $|O(1)|E_{O(1)}$ has eigenvalues $(1, 4, 6, 3)$.

For the case $c = 2$, observe that $\alpha(2)$ defines the coweight sum $p_3(\mathfrak{B}_1) + p_6(\mathfrak{B}_3)$ in $\bar{\mathfrak{h}}$. We have

$$\frac{\| \pi_p(\alpha(2)) \|^2}{\| \alpha(2) \|^2} = \frac{1}{6}, \quad \frac{\| p_3(\mathfrak{B}_1) \|^2}{\| \alpha(2) \|^2} = \frac{1}{3}, \quad \frac{\| p_6(\mathfrak{B}_3) \|^2}{\| \alpha(2) \|^2} = \frac{1}{2}$$

(summing to 1). So the eigenvalues of $E_{O(2)}$ are $(\frac{1}{6}, \frac{1}{6}, 0, \frac{1}{6})$. It's easy to see that $|O(2)| = 3 \cdot 6 = 18$ and so $2|O(2)|E_{O(2)}$ has eigenvalues $(6, 6, 0, 6)$.

For the case $c = 3$, observe that $\alpha(3)$ defines the coweight sum $p_2(\mathfrak{B}_2) + p_7(\mathfrak{B}_3)$ in $\bar{\mathfrak{h}}$. We have

$$\frac{\| \pi_p(\alpha(3)) \|^2}{\| \alpha(3) \|^2} = \frac{3}{8}, \quad \frac{\| p_2(\mathfrak{B}_2) \|^2}{\| \alpha(3) \|^2} = \frac{1}{4}, \quad \frac{\| p_7(\mathfrak{B}_3) \|^2}{\| \alpha(3) \|^2} = \frac{3}{8}$$

(summing to 1). So the eigenvalues of $E_{O(3)}$ are $(\frac{3}{8}, 0, \frac{1}{4}, \frac{1}{8})$. It's easy to see that $|O(3)| = 2 \cdot 4 = 8$ and so $3|O(3)|E_{O(3)}$ has eigenvalues $(9, 0, 6, 3)$.

For the case $c = 4$, observe that $\alpha(4)$ defines the coweight sum $p_1(\mathfrak{B}_1)$ in $\bar{\mathfrak{h}}$. We have

$$\frac{\| \pi_p(\alpha(4)) \|^2}{\| \alpha(4) \|^2} = \frac{2}{3}, \quad \frac{\| p_1(\mathfrak{B}_1) \|^2}{\| \alpha(4) \|^2} = \frac{1}{3}$$

(summing to 1). So the eigenvalues of $E_{O(4)}$ are $(\frac{2}{3}, \frac{1}{6}, 0, 0)$. It's easy to see that $|O(4)| = 3$ and so $4|O(4)|E_{O(4)}$ has eigenvalues $(8, 2, 0, 0)$.

We conclude that $U_p = -kE = -k\sum_{c=1}^{4} c|O(c)|E_{O(c)}$ has as eigenvalues the system $(-24k, -12k, -12k, -12k)$. In particular, $a(p,p) = 144k^2$.

Using this way, we have the eigenvalues for type E_n as follows:

E_6		
p	eigenvalues	multiplicities
$\varpi_1^\vee$,$\varpi_6^\vee$	$(-4k,-2k)$	$(1,6)$
$\varpi_2^\vee$	$(-4k,-3k)$	$(1,6)$
$\varpi_3^\vee$,$\varpi_5^\vee$	$(-5k,-4k)$	$(2,5)$
$\varpi_4^\vee$	$-6k$	7

E_7		
p	eigenvalues	multiplicities
$\varpi_1^\vee$	$(-6k,-4k)$	$(1,7)$
$\varpi_2^\vee$	$(-7k,-6k)$	$(1,7)$
$\varpi_3^\vee$	$(-9k,-8k)$	$(2,6)$
$\varpi_4^\vee$	$-12k$	8
$\varpi_5^\vee$	$(-9k,-10k)$	$(5,3)$
$\varpi_6^\vee$	$(-6k,-8k)$	$(6,2)$
$\varpi_7^\vee$	$(-3k,-6k)$	$(7,1)$

E_8		
p	eigenvalues	multiplicities
$\varpi_1^\vee$	$(-12k,-10k)$	$(1,8)$
$\varpi_2^\vee$	$(-16k,-15k)$	$(1,8)$
$\varpi_3^\vee$	$(-21k,-20k)$	$(2,7)$
$\varpi_4^\vee$	$-30k$	9
$\varpi_5^\vee$	$(-24k,-25k)$	$(5,4)$
$\varpi_6^\vee$	$(-18k,-20k)$	$(6,3)$
$\varpi_7^\vee$	$(-12k,-15k)$	$(7,2)$
$\varpi_8^\vee$	$(-6k,-10k)$	$(8,1)$

These computation show that there are at most 2 eigenvalues along the toric divisor for any root system.

Remark 3.11. We notice from the above computation that if we extend the bilinear symmetric form a on $\mathfrak{h}$ to the space $\mathfrak{h}\oplus\mathbb{C}$ by the way such that

$$\begin{cases} a(q,t\dfrac{\partial}{\partial t})=0 \quad \text{for} \quad \forall q\in\mathfrak{h} \\ a(t\dfrac{\partial}{\partial t},t\dfrac{\partial}{\partial t})=-1, \end{cases}$$

the two eigenvectors in the space which is spanned by the fundamental coweight p and $t\frac{\partial}{\partial t}$ are perpendicular to each other with respect to this a. This also gives rise to a Frobenius algebra on $H^\circ\times\mathbb{C}^\times$ which will be investigated in detail in Chapter 4.

We also have the dilatation field as follows.

Theorem 3.12. *Suppose an affine structure on $H^\circ \times \mathbb{C}^\times$ is given by the torsion free flat connection $\tilde{\nabla}$ defined by (2.2), then the vector field $t\frac{\partial}{\partial t}$ is in fact a dilatation field on $H^\circ \times \mathbb{C}^\times$ with factor $\lambda = 1$.*

Proof. It's a straightforward computation. Suppose a local vector field $\tilde{v}$ on $H^\circ \times \mathbb{C}^\times$ is of the form $\tilde{v} := v + \mu t\frac{\partial}{\partial t}$ where v is a vector field on H°, we have

$$\begin{aligned}\tilde{\nabla}_{\tilde{v}}(t\frac{\partial}{\partial t}) &= \tilde{\nabla}^0_{v+\mu t\frac{\partial}{\partial t}}(t\frac{\partial}{\partial t}) - \tilde{\Omega}^*_{v+\mu t\frac{\partial}{\partial t}}(t\frac{\partial}{\partial t}) \\ &= 0 - 0 - 0 - 0 + \sum_{\alpha_i \in \mathfrak{B}} \alpha_i(v)\partial_{p_i} + \mu t\frac{\partial}{\partial t} \\ &= \tilde{v}\end{aligned}$$

since $t\frac{\partial}{\partial t}$ is flat with respect to $\tilde{\nabla}^0$. □

We then could decompose the vector bundle V (with its flat connection) naturally according to these eigenspaces.

Lemma 3.13. *The vector bundle V (with its flat connection) decomposes naturally according to the images in $\mathbb{C}/\mathbb{Z}$ of the eigenvalues of the residue endomorphism: $V = \bigoplus_{\zeta \in \mathbb{C}^\times} V^\zeta$, where V^ζ has a residue endomorphism whose eigenvalue ν are such that $\exp(2\pi\sqrt{-1}\nu) = \zeta$.*

And suppose D is a hypersurface in M. Then the affine structure on $M - D$ has a degeneration along D° which could be decomposed naturally as above with logarithmic exponent $\nu - 1$ in each corresponding eigenspace.

Proof. [**6**]. □

3.3. Reflection representation

We recall that $R \subset \mathfrak{a}^*$ is a reduced irreducible finite root system where $\mathfrak{a}^*$ is a Euclidean vector space of dimension n. Let $\mathfrak{a} = \mathrm{Hom}(\mathfrak{a}^*, \mathbb{R})$ be the dual Euclidean vector space, and let $R^\vee$ in $\mathfrak{a}$ be the dual root system and denote the corresponding coroot lattice by $Q^\vee = \mathbb{Z}R^\vee$. We then have the weight lattice $P = \mathrm{Hom}(Q^\vee, \mathbb{Z})$ of R in $\mathfrak{a}^*$. The torus having P as (rational) character lattice, $H' = \mathrm{Hom}(P, \mathbb{C}^\times)$ is often called the simply connected torus. Put $H'^\circ = H' - \cup_{\alpha \in R_+} H'_\alpha$ where $H'_\alpha = \{h \in H' \mid e^\alpha(h) = 1\}$. Let

$$C = \{h \in H' \mid e^\alpha(h) = 1 \text{ for all } \alpha \in R\} \cong P^\vee/Q^\vee,$$

so the adjoint torus H is just H'/C. Then as discussed in Theorem 2.8 the special differential equation system (2.3) associated with the root system R

gives a W'-invariant projective structure on H'° where $W' = W \rtimes C$ is the extended Weyl group.

In the example of the root system of type A_1 this equation boils down to (take $u = v = \alpha^\vee/2$, $k' = 0$ with variable $z = e^\alpha(h)$ and derivative $\theta = z\partial$)

$$(\theta^2 + k\frac{1+z^{-1}}{1-z^{-1}}\theta + \frac{1}{4}k^2)f(z) = 0.$$

We shall now construct the reflection representation of the affine Artin group $\mathrm{Art}(M)$ with generators $\sigma_0, \cdots \sigma_n$ and braid relations

$$\underbrace{\sigma_i\sigma_j\sigma_i\cdots}_{m_{ij}} = \underbrace{\sigma_j\sigma_i\sigma_j\cdots}_{m_{ij}}$$

for all $i \neq j$ where both members are words comprising m_{ij} letters. These results can be traced back to Coxeter and Kilmoyer, see e.g. [**8**] and [**9**].

First we need to investigate what the two complex reflections look like if they satisfy a braid relation.

Proposition 3.14. *Let $s_1, s_2 \in \mathrm{GL}_2(\mathbb{C})$ be the complex reflections as follows.*

$$\begin{pmatrix} -q_1 & d_1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ d_2 & -q_2 \end{pmatrix}$$

where $q_1, q_2 \in \mathbb{C}^\times$.

If $m = 2r+1$ $(r \geq 1)$ is odd, then s_1 and s_2 satisfy the braid relation of length m

$$(s_1s_2)^r s_1 = (s_2s_1)^r s_2$$

if and only if

$$q_1 = q_2 (= q \textit{ say}),\ d_1d_2 = (2 + \xi + \xi^{-1})q \textit{ with } \xi^m = 1, \xi \neq 1.$$

If $m = 2r$ $(r \geq 1)$ is even, then s_1 and s_2 satisfy the braid relation of length m

$$(s_1s_2)^r = (s_2s_1)^r$$

if and only if

$$d_1 = d_2 = 0 \textit{ for } m = 2;$$

and

$$d_1d_2 = q_1 + q_2 + (\xi + \xi^{-1})q_1^{\frac{1}{2}}q_2^{\frac{1}{2}} \textit{ with } \xi^m = 1, \xi^2 \neq 1 \textit{ for } m \geq 4.$$

Proof. It is obvious that $\det(s_1) = -q_1$ and $\det(s_2) = -q_2$.

Let us do the odd case first. Suppose $m = 2r+1$ is odd. Put $T_1 = (s_1s_2)^r s_1$ and $T_2 = (s_2s_1)^r s_2$. Suppose the braid relation holds, i.e., $T_1 = T_2$ ($=T$ say). It is clear that $d_1d_2 \neq 0$, for otherwise it contradicts the braid relation. So the group generated by s_1 and s_2 acts irreducibly on $\mathbb{C}^2$. We also have that $Ts_1 = s_2T$ and $Ts_2 = s_1T$ which follows that s_1 and s_2 are conjugated, hence we have $q_1 = q_2$ ($=q$ say). Moreover, $T^2 = (s_1s_2)^m = (s_2s_1)^m$ commutes with both s_1 and s_2, and hence is a scalar matrix by Schur's lemma, i.e., $T^2 - q^mI = 0$. But T is not a scalar matrix, so it has eigenvalues $\pm\lambda$ with $\lambda^2 = q^m$ since $\det(T) = -q^m$. Hence s_1s_2 has eigenvalues $\xi q, \xi^{-1}q$ with $\xi^m = 1, \xi \neq 1$. Since we have $\mathrm{tr}(s_1s_2) = d_1d_2 - q_1 - q_2$, we have also that $d_1d_2 = (2+\xi+\xi^{-1})q$ with $\xi^m = 1, \xi \neq 1$.

Conversely, suppose $q_1 = q_2$ ($= q$ say), $d_1d_2 = (2+\xi+\xi^{-1})q$ with $\xi^m = 1.\xi \neq 1$. Then $\det(s_1s_2) = \det(s_2s_1) = q^2$ and $\mathrm{tr}(s_1s_2) = \xi q + \xi^{-1}q$, so s_1s_2 and s_2s_1 have eigenvalues $\xi q, \xi^{-1}q$ with $\xi^m = 1, \xi \neq 1$. Hence $T_1T_2 = T_2T_1 = q^mI$. We notice that the matrix $\begin{pmatrix} 0 & d_1 \\ d_2 & 0 \end{pmatrix}$ conjugates s_1 to s_2, and therefore also T_1 to T_2. If T_1 and T_2 have eigenvalues λ_1, λ_2, then $\lambda_1\lambda_2 = \det(T_1) = -q^m$. We also have

$$\lambda_1 + \lambda_2 = \mathrm{tr}(T_1) = \mathrm{tr}(T_2) = q^m(\lambda_1^{-1} + \lambda_2^{-1}) = -(\lambda_1 + \lambda_2)$$

so $\lambda_1 = -\lambda_2$. In turn this implies $T_1^2 = q^mI$ and hence the braid relation $T_1 = T_2$ follows.

Now suppose $m = 2r$ ($r \geq 1$) is even. In case $m = 2$ it is direct to verify that the relation $s_1s_2 = s_2s_1$ holds if and only if $d_1 = d_2 = 0$. Therefore assume $r \geq 2$ now. Then the group generated by s_1 and s_2 acts irreducibly on $\mathbb{C}^2$. Put $T_1 = (s_1s_2)^r$ and $T_2 = (s_2s_1)^r$. The braid relation $T_1 = T_2$ ($= T$ say) implies that T commutes with s_1 and s_2, and so T is a scalar matrix, λI say, with $\lambda^2 = (q_1q_2)^r$. Hence the matrix s_1s_2 has eigenvalues $\xi^{\pm1}q_1^{\frac{1}{2}}q_2^{\frac{1}{2}}$ with $\xi^m = 1, \xi^2 \neq 1$. The trace computation of s_1s_2 shows that $d_1d_2 = q_1 + q_2 + (\xi+\xi^{-1})q_1^{\frac{1}{2}}q_2^{\frac{1}{2}}$.

Conversely, suppose the above equality holds. Then the eigenvalues of s_1s_2 and s_2s_1 are $\xi^{\pm1}q_1^{\frac{1}{2}}q_2^{\frac{1}{2}}$, and $T_1 = T_2 = \xi^r(q_1^{\frac{1}{2}}q_2^{\frac{1}{2}})^rI$. □

In fact, finite complex reflection groups are already classified by Shephard and Todd in 1954 [**32**].

Let $M = (m_{ij})_{0\leq i,j\leq n}$ be the affine Coxeter matrix associated with the extended Dynkin diagram of the affine root system $\tilde{R}$ associated with R. If $\alpha_1, \cdots \alpha_n$ are the simple roots in R_+, then $a_0 = -\tilde{\alpha}, a_1 = \alpha_1, \cdots, a_n = \alpha_n$ are the simple roots in $\tilde{R}_+$ with $\tilde{\alpha}$ the highest root in R_+.

Recall that K is the space of multiplicity parameters for R defined by

$$\kappa = (k_\alpha)_{\alpha\in R} \in \mathbb{C}^R$$

where κ is invariant on W-orbits in R. It is clear that K is isomorphic to $\mathbb{C}^r$ as a $\mathbb{C}$-vector space if r is the number of W-orbits in R (i.e., $r = 1$ or 2). Like in Chapter 2, we shall still sometimes write k_i instead of k_{α_i} if $\{\alpha_1, \cdots, \alpha_n\}$ is a basis of simple roots in R_+. And sometimes we also write k for k_1 and

k' for k_n if $\alpha_n \notin W\alpha_1$ when no confusion can arise. But note that k' has a different meaning for type A_n, which can be seen from Remark 2.5.

For R of any type other than A_n, let $q_i^{\frac{1}{2}}$ and $s_{ij} = s_{ji}$ for $0 \le i \ne j \le n$ be indeterminants with additional relations

$$\begin{cases} s_{ij} = 0 & \text{if} \quad m_{ij} = 2 \\ q_i^{\frac{1}{2}} = q_j^{\frac{1}{2}} \text{ and } s_{ij} = 1 & \text{if} \quad m_{ij} = 3 \\ s_{ij}^2 = q_i^{\frac{1}{2}} q_j^{-\frac{1}{2}} + q_i^{-\frac{1}{2}} q_j^{\frac{1}{2}} + 2\cos(\frac{2\pi}{m_{ij}}) & \text{if} \quad m_{ij} \ge 4 \end{cases}$$

and let $\mathcal{A}$ be the ring $\mathbb{Z}[q_i^{\frac{1}{2}}, q_i^{-\frac{1}{2}}, s_{ij} \mid 0 \le i \ne j \le n]$. Let $^-$ be the involution of $\mathcal{A}$ defined by $\bar{q}_i^{\frac{1}{2}} = q_i^{-\frac{1}{2}}$ and $\bar{s}_{ij} = s_{ij}$. Let K' be the space of restricted multiplicity parameters defined by

$$K' = \{\kappa = (k_{\alpha_i}) \in K \mid k_i \in (-\frac{1}{2}, \frac{1}{2}), |k_i - k_j| < 1 - \frac{2}{m_{ij}} \text{ if } m_{ij} \ge 4 \text{ and even}\}.$$

For $z \in \mathbb{C} \setminus (-\infty, 0]$, let $z^{\frac{1}{2}}$ denote the branch of the square root with $1^{\frac{1}{2}} = 1$. If $\kappa \in K'$ is a restricted multiplicity parameter on R then the substitutions

$$q_j^{\frac{1}{2}} = \exp(-\pi\sqrt{-1}k_j), \quad s_{ij} = (2\cos\pi(k_i - k_j) + 2\cos(\frac{2\pi}{m_{ij}}))^{\frac{1}{2}}$$

for all j and $i \ne j$ with $m_{ij} \ge 3$ induce a homomorphism $\mathcal{A} \to \mathbb{C}$ called specialization of $\mathcal{A}$ at $\kappa \in K'$.

The root system R of type A_n is somewhat peculiar due to the fact that the extended Dynkin diagram is a cycle rather than a tree. For R of type A_n, we take

$$s_{0,1} = \cdots = s_{n-1,n} = s_{n,0} = q'^{-\frac{1}{2}}, \qquad s_{1,0} = \cdots = s_{n,n-1} = s_{0,n} = q'^{\frac{1}{2}}$$

and put $\bar{q}^{\frac{1}{2}} = q^{-\frac{1}{2}}$, $\bar{q}'^{\frac{1}{2}} = q'^{-\frac{1}{2}}$ (with $q^{\frac{1}{2}} = q_i^{\frac{1}{2}}$ for $i = 0, \cdots, n$). Let the restricted parameter space K' be defined by

$$K' = \{(k, k') \mid k \in (-\frac{1}{2}, \frac{1}{2}), k' \in (-\frac{1}{2}, \frac{1}{2})\}$$

and let the specialization of $\mathcal{A}$ at $\kappa \in K'$ be given by the substitutions

$$q^{\frac{1}{2}} = \exp(-\pi\sqrt{-1}k), \quad q'^{\frac{1}{2}} = \exp(-\pi\sqrt{-1}k').$$

Remark 3.15. In all cases the involution $^-$ of $\mathcal{A}$ becomes complex conjugation under specialization.

Now let $e_i (i = 0, \cdots, n)$ be the standard basis of $\mathcal{A}^{n+1}$ and define a Hermitian form on $\mathcal{A}^{n+1}$ with Gram matrix of the standard basis given by

$$h_{ij} = \begin{cases} q_i^{\frac{1}{2}} + q_i^{-\frac{1}{2}} & \text{if} \quad i = j \\ -s_{ij} & \text{if} \quad i \neq j \end{cases} \tag{3.1}$$

Indeed we have $h^\dagger = h$ with $h^\dagger = \bar{h}^t$. The unitary reflection T_j of $\mathcal{A}^{n+1}$ having e_j as eigenvector with eigenvalue $-q_j$ satisfies

$$T_j(e_i) = e_i - q_i^{\frac{1}{2}} h_{ij} e_j \quad \text{for all } i.$$

From this identity we can see that T_j also satisfies the quadratic relation

$$(T_j - 1)(T_j + q_j) = 0$$

as well as the braid relation

$$\underbrace{T_i T_j T_i \cdots}_{m_{ij}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij}}$$

for all $i \neq j$. Therefore there exists a unique (unitary) representation

$$\begin{aligned} \rho : \mathrm{Art}(M) &\to \mathrm{GL}_{n+1}(\mathcal{A}) \\ \sigma_j &\mapsto T_j \end{aligned} \tag{3.2}$$

and this is the reflection representation of $\mathrm{Art}(M)$.

The special hypergeometric system (2.3) in Theorem 2.8 is defined on the complex manifold H° and is invariant for W. Therefore it lives on the complex orbifold $W \backslash H^\circ$ as well.

This in turn implies that the monodromy of the system (2.3) is a homomorphism:

$$\rho' : \pi_1^{\mathrm{orb}}(W \backslash H^\circ, W p_0) \to \mathrm{GL}_{n+1}(\mathbb{C}) \tag{3.3}$$

with $p_0 \in D_+ \subset H^\circ$ a fixed base point where D_+ is the fundamental alcove of H°.

Remark 3.16. There is a natural isomorphism using Brieskorn's theorem [**3**]

$$\pi_1(W \backslash H'^\circ, W' p_0) \cong \mathrm{Art}(M)$$

and then there is a corresponding isomorphism

$$\pi_1^{\mathrm{orb}}(W \backslash H^\circ, W p_0) \cong \mathrm{Art}(M) \rtimes C$$

with C viewed as group of diagram automorphisms of the extended Dynkin diagram of R and hence acting naturally on the generators of $\mathrm{Art}(M)$. The reflection representation (3.2) can be extended in a natural way to a representation of $\mathrm{Art}(M) \rtimes C$. We write $\mathrm{Art}'(M) = \mathrm{Art}(M) \rtimes C$. If $\mathrm{Aut}(M)$ denotes the full group of diagram automorphisms of the extended Dynkin diagram, then the reflection representation (3.2) extends in a natural way to a representation of $\mathrm{Art}(M) \rtimes \mathrm{Aut}(M)$ with the exception of type A_n. This extension for type A_n is only possible when $q' = 1$.

Then we identify the two representations as follows.

Theorem 3.17. *The monodromy representation of the special hypergeometric system given in Theorem 2.8 for a parameter $\kappa \in K'$ is equal to the specialization at $\kappa \in K'$ of the reflection representation of the affine Artin group. In particular for $\kappa \in K'$ the local solution space at $p_0 \in D_+$ admits a Hermitian form $h(\kappa)$ invariant under monodromy.*

PROOF. From the computation on the eigenvalues of the residue endomorphisms of the connection along the mirrors, the special hypergeometric system has exponents along the wall of D_+ corresponding to the simple root α_j equal to 1 and $1 - 2k_j$ with multiplicities n and 1 respectively. Therefore the monodromy $\rho'(g_j)$ of the special hypergeometric system corresponding to a half turn around that wall is a complex reflection with a quadratic relation

$$(\rho'(g_j) - 1)(\rho'(g_j) + q_j) = 0$$

with $q_j = \exp(-2\pi\sqrt{-1}k_j)$, which coincide with the relation for the reflection representation of $\mathrm{Art}(M)$ when the κ is the same. Moreover, for $\kappa \in K'$ being generic, the monodromy is easily seen to be irreducible. The result hence follows if for type A_n, k' and q' are related by the specialization $q' = \exp(-2\pi\sqrt{-1}k')$. □

Theorem 3.18. *The specialization* $\det(h(\kappa))$ *at* $\kappa \in K'$ *is given for type* $ABCFG$ *by*

$$\det(h(\kappa)) = -4\sin(\pi x)\sin(\pi y), \tag{3.4}$$

with $(x, y) = ((n+1)(k+k')/2, (n+1)(k-k')/2), ((n-2)k+k', 2k), ((n-2)k+2k', k), (k+k', 2k+k'), ((k+3k')/2, (k+k')/2)$ *respectively,*

and for type D_n and E_n by

$$\det(h(\kappa)) = 2^{n+1}\prod_{j=0}^{n}(\cos(\pi k) - \cos(\pi\tilde{m}_j/\tilde{h})) \tag{3.5}$$

with $\tilde{h}$ and $\{\tilde{m}_j\}$ given by

$$\begin{aligned}
&D_n : \tilde{h} = 2(n-2), && \{\tilde{m}_j\} = \{0, 2, \cdots, 2(n-2), (n-2), (n-2)\}\\
&E_6 : \tilde{h} = 6, && \{\tilde{m}_j\} = \{0, 2, 2, 3, 4, 4, 6\}\\
&E_7 : \tilde{h} = 12, && \{\tilde{m}_j\} = \{0, 3, 4, 6, 6, 8, 9, 12\}\\
&E_8 : \tilde{h} = 30, && \{\tilde{m}_j\} = \{0, 6, 10, 12, 15, 18, 20, 24, 30\}
\end{aligned}$$

Proof. The first identity is obtained directly from the classification theory of connected extended Dynkin diagrams, which is explained in detail in Appendix A. And the second identity appears as Exercise 4 of Ch. V, § 6 in Bourbaki [**2**]. □

Corollary 3.19. *For R of type $ABCFG$ put*

$$K'_{\mathrm{hyp}} = \{\kappa \in K' \mid 0 < x < 1, 0 < y < 1\}$$

and for type DE put

$$K'_{\mathrm{hyp}} = (0, \frac{1}{n-2}) \quad \textit{and} \quad (0, \frac{1}{n-3})$$

respectively. Then the monodromy representation has an invariant Hermitian form of Lorentz signature (n,1).

Proof. Observe that for $k \in \sqrt{-1}\mathbb{R}^\times$, $k' = 0$ (for R of type A) and for $k = k' \in \sqrt{-1}\mathbb{R}^\times$ (for R of other types), the form $h(\kappa)$ is positive definite, and for $\kappa \in K'_{\mathrm{hyp}}$, one has $\det(h(\kappa)) < 0$. Since on the line $k' = 0$ (type A) and $k = k'$ (other types) the function $\det(h(\kappa))$ has a double zero at the origin, the result follows. □

3.4. The evaluation map

Recall that $D^\kappa_{u,v}$ denotes the second order differential operator in (2.3):

$$\partial_u\partial_v + \frac{1}{2}\sum_{\alpha>0} k_\alpha \alpha(u)\alpha(v)\frac{e^\alpha+1}{e^\alpha-1}\partial_{\alpha^\vee} + \partial_{b^\kappa(u,v)} + a^\kappa(u,v).$$

Take $p \in H^\circ$ and $\kappa \in K$. Consider $D^\kappa_{u,v}$ as an operator on the stalk of holomorphic germs $\mathcal{O}_{\kappa,p}$. Then the solutions form a free $\mathcal{O}_\kappa$ module of rank $n+1$. Hence the local solutions of $D^\kappa_{u,v}f = 0$ for $\forall u, v \in \mathfrak{h}$ near p can be considered as a vector bundle $\mathcal{F}_p$ over K. Any $w \in W$ induces an isomorphism of vector bundle $\mathcal{F}_p$ and $\mathcal{F}_{wp}$. Then we can identify all these vector bundles induced by a regular W-orbit S. This yields a vector bundle $\mathcal{F}_S$ over K of rank $n+1$, the fiber of which is denoted by $\mathcal{F}_S(\kappa)$. According to the preceding section, we have the following representation $\rho(\kappa)$ on the vector bundle $\mathcal{F}_S(\kappa)$ by specializing κ:

$$\rho : \pi_1^{\mathrm{orb}}(W\backslash H^\circ, S) \cong \mathrm{Art}'(M) \to \mathrm{End}(\mathcal{F}_S).$$

Transposing ρ yields a following representation

$$\rho^* : \mathrm{Art}'(M) \to \mathrm{End}(\mathcal{F}_S^*).$$

Let $h^*(\kappa)$ be given as follows:

$$h^*(\kappa) = \det(h(\kappa))(h(\kappa))^{-1} \tag{3.6}$$

We then have the dual Hermitian form $h^*(\kappa)$ of $h(\kappa)$ for the transpose $\rho^*(\kappa)$ if $\kappa \in K'$ is real valued.

Lemma 3.20. *The dual Hermitian form $h^*(\kappa)$ of $h(\kappa)$ given as above is a nontrivial invariant Hermitian form for the transpose $\rho^*(\kappa)$ if $\kappa \in K'$ is real valued. Moreover, if $h(\kappa)$ is positive definite, then $h^*(\kappa)$ is also positive definite; if $h(\kappa)$ is parabolic, then $h^*(\kappa)$ is positive semidefinite with n dimensional kernel; and if $h(\kappa)$ is hyperbolic, then $h^*(\kappa)$ is of the signature $(1, n)$.*

Proof. It's clear that $h^*(\kappa)$ is a nontrivial invariant Hermitian form for $\rho^*(\kappa)$ since $h(\kappa)$ is of rank at least n and then the matrix $h^*(\kappa)$ is of rank at least 1. In fact, $h^*(\kappa)$ is just the minor matrix of $h(\kappa)$ and we have $h^*(\kappa)h(\kappa) = \det(h(\kappa))I$, the statement easily follows. $\square$

Since the differential operator $D^{\kappa}_{u,v}$ defines an affine structure on $H^\circ \times \mathbb{C}^\times$, then the locally affine-linear functions which are of the form $c + tf$ by Lemma 2.2 make up a local system $\mathrm{Aff}_{H^\circ\times\mathbb{C}^\times}$ of $\mathbb{C}$-vector space in the structure sheaf $\mathcal{O}_{H^\circ\times\mathbb{C}^\times}$. This local system is of rank $n+2$ and contains the constants $\mathbb{C}_{H^\circ\times\mathbb{C}^\times}$. Then the quotient $\mathrm{Aff}_{H^\circ\times\mathbb{C}^\times}/\mathbb{C}_{H^\circ\times\mathbb{C}^\times}$ is a local system whose underlying vector bundle is the cotangent bundle of $H^\circ \times \mathbb{C}^\times$. So there exists a multivalued evaluation map given by

$$ev : K \times ((W\backslash H^\circ) \times \mathbb{C}^\times) \dashrightarrow \mathcal{F}^*_S$$

such that $ev(\kappa, p, t)(f) = tf(p)$, and a corresponding projective evaluation map

$$Pev : K \times (W\backslash H^\circ) \dashrightarrow \mathbb{P}(\mathcal{F}^*_S).$$

In order to eliminate the multivaluedness of this map, let us denote by $\widehat{W\backslash H^\circ}$ the universal Γ-covering space of $W\backslash H^\circ$ with $\Gamma = \pi_1(W\backslash H^\circ)/\mathrm{Ker}(Pr \circ \rho)$ the projective monodromy group. Here we write $Pr : \mathrm{GL}(\mathcal{F}_S(\kappa)) \twoheadrightarrow \mathrm{PGL}(\mathcal{F}_S(\kappa))$ for the natural map. In other words, $\widehat{W\backslash H^\circ}$ is equal to $\mathrm{Ker}(Pr \circ \rho)\backslash(\widetilde{W\backslash H^\circ})$ with $\widetilde{W\backslash H^\circ}$ the universal covering of $W\backslash H^\circ$. Then we have the commutative diagram

$$\begin{array}{ccc}
\widehat{W\backslash H^\circ} & \xrightarrow{\widehat{Pev}} & \mathbb{P}(\mathcal{F}^*_S(\kappa)) \\
\Big\downarrow & & \Big\downarrow \\
W\backslash H^\circ & \xrightarrow{Pev} & \Gamma\backslash\mathbb{P}(\mathcal{F}^*_S(\kappa))
\end{array}$$

Note that $\Gamma\backslash\mathbb{P}(\mathcal{F}^*_S)$ is an ill defined space unless the action of Γ on $\mathbb{P}(\mathcal{F}^*_S)$ is properly discontinuous.

Recall that the Wronskian of $D^{\kappa}_{u,v}$ is defined up to a scalar multiplication as follows:

$$J := \det(\partial_{\xi_i} f_j)_{0\le i,j\le n}$$

where $\xi_1, \cdots, \xi_n \in \mathfrak{h}$ being an orthonormal basis and let $f_0, \cdots, f_n$ be a basis of local solutions of $D^{\kappa}_{u,v} f = 0$.

Lemma 3.21. *The Wronskian of $D^{\kappa}_{u,v}$ is given by:*

$$J = \prod_{\alpha>0} (e^{\alpha/2} - e^{-\alpha/2})^{-2k_\alpha}.$$

Proof. First we compute

$$\begin{aligned}
\partial_\xi J =& \partial_\xi \det \begin{pmatrix} f_0 & \partial_{\xi_1} f_0 & \cdots & \partial_{\xi_n} f_0 \\ f_1 & \partial_{\xi_1} f_1 & \cdots & \partial_{\xi_n} f_1 \\ \vdots & \vdots & \ddots & \vdots \\ f_n & \partial_{\xi_1} f_n & \cdots & \partial_{\xi_n} f_n \end{pmatrix} \\
=& \det \begin{pmatrix} \partial_\xi f_0 & \partial_{\xi_1} f_0 & \cdots & \partial_{\xi_n} f_0 \\ \partial_\xi f_1 & \partial_{\xi_1} f_1 & \cdots & \partial_{\xi_n} f_1 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_\xi f_n & \partial_{\xi_1} f_n & \cdots & \partial_{\xi_n} f_n \end{pmatrix} \\
& + \sum_i \det \begin{pmatrix} f_0 & \cdots & \partial_\xi \partial_{\xi_i} f_0 & \cdots & \partial_{\xi_n} f_0 \\ f_1 & \cdots & \partial_\xi \partial_{\xi_i} f_1 & \cdots & \partial_{\xi_n} f_1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ f_n & \cdots & \partial_\xi \partial_{\xi_i} f_n & \cdots & \partial_{\xi_n} f_n \end{pmatrix} \\
=& \sum_i \det \begin{pmatrix} f_0 & \cdots & \partial_\xi \partial_{\xi_i} f_0 & \cdots & \partial_{\xi_n} f_0 \\ f_1 & \cdots & \partial_\xi \partial_{\xi_i} f_1 & \cdots & \partial_{\xi_n} f_1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ f_n & \cdots & \partial_\xi \partial_{\xi_i} f_n & \cdots & \partial_{\xi_n} f_n \end{pmatrix}
\end{aligned}$$

while because of (2.3), for type A_n, we have

$$\begin{aligned}
&\partial_\xi \partial_{\xi_i} f_j \\
&= -\frac{1}{2} \sum_{\alpha>0} k_\alpha \frac{e^\alpha + 1}{e^\alpha - 1} \alpha(\xi)\alpha(\xi_i) \partial_{\alpha^\vee} f_j - \partial_{b^\kappa(\xi,\xi_i)} f_j - a^\kappa(\xi,\xi_i) f_j \\
&= -\frac{1}{2} \sum_{\alpha>0} k_\alpha \frac{e^\alpha + 1}{e^\alpha - 1} \alpha(\xi)\alpha(\xi_i) \partial_{\alpha^\vee} f_j - \frac{1}{2} k' \sum_{\alpha>0} \alpha(\xi)\alpha(\xi_i) \partial_{\alpha'} f_j - a^\kappa(\xi,\xi_i) f_j \\
&= -\frac{1}{2} \sum_{\alpha>0} \sum_i k_\alpha \frac{e^\alpha + 1}{e^\alpha - 1} \alpha(\xi)\alpha(\xi_i)(\alpha^\vee, \xi_i) \partial_{\xi_i} f_j \\
&\quad -\frac{1}{2} k' \sum_{\alpha>0} \sum_i \alpha(\xi)\alpha(\xi_i)(\alpha', \xi_i) \partial_{\xi_i} f_j - a^\kappa(\xi,\xi_i) f_j;
\end{aligned}$$

and then

$$\begin{aligned}
&\partial_\xi J \\
=&\sum_i \Big(-\frac{1}{2}\sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1}\alpha(\xi)\alpha(\xi_i)(\alpha^\vee,\xi_i)\Big)J + \sum_i \Big(-\frac{1}{2}k' \sum_{\alpha>0} \alpha(\xi)\alpha(\xi_i)(\alpha',\xi_i)\Big)J \\
=&-\sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1}\alpha(\xi)J.
\end{aligned}$$

For other types, we have

$$\begin{aligned}
\partial_\xi \partial_{\xi_i} f_j &= -\frac{1}{2}\sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1}\alpha(\xi)\alpha(\xi_i)\partial_{\alpha^\vee} f_j - a^\kappa(\xi,\xi_i) f_j \\
&= -\frac{1}{2}\sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1}\alpha(\xi)\alpha(\xi_i)(\alpha^\vee,\xi_i)\partial_{\xi_i} f_j - a^\kappa(\xi,\xi_i) f_j;
\end{aligned}$$

and then

$$\begin{aligned}
\partial_\xi J &= \sum_i \Big(-\frac{1}{2}\sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1}\alpha(\xi)\alpha(\xi_i)(\alpha^\vee,\xi_i)\Big)J \\
&= -\sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1}\alpha(\xi)J.
\end{aligned}$$

Then we verify that the proposed product formula for J satisfies all these formulas.

$$\begin{aligned}
\partial_\xi J =& \sum_{\alpha>0} \Big(-2k_\alpha (e^{\alpha/2}-e^{-\alpha/2})^{-2k_\alpha-1}\cdot(e^{\alpha/2}\cdot\frac{\alpha(\xi)}{2}+e^{-\alpha/2}\cdot\frac{\alpha(\xi)}{2}) \\
&\cdot \prod_{\substack{\beta\neq\alpha\\ \beta>0}} (e^{\beta/2}-e^{-\beta/2})^{-2k_\beta}\Big) \\
=& -\sum_{\alpha>0}\Big(k_\alpha \alpha(\xi)\frac{e^\alpha+1}{e^\alpha-1}\prod_{\beta>0}(e^{\beta/2}-e^{-\beta/2})^{-2k_\beta}\Big) \\
=& -\sum_{\alpha>0} k_\alpha \alpha(\xi)\frac{e^\alpha+1}{e^\alpha-1}J
\end{aligned}$$

The lemma follows. □

A basis $f_0,\cdots,f_n$ of $\mathcal{F}_S(\kappa)$ identifies $\mathbb{P}(\mathcal{F}_S^*(\kappa))$ with $\mathbb{P}^n(\mathbb{C})$ by the following way

$$\begin{aligned}
Pev(\kappa): \widehat{W\backslash H^\circ} &\to \mathbb{P}^n(\mathbb{C}) \\
q &\mapsto [f_0(q):f_1(q):\cdots:f_n(q)]
\end{aligned}$$

Since an irreducible component of $\hat{H} - H^\circ$ is either the closure $\hat{H}_\alpha$ in $\hat{H}$ of some H_α or is equal to some D_p with $p \in \Pi$. Let $I \subset \{1, 2, \cdots, n\}$ have m elements. The subset

$$\{h \in \hat{H} \mid e^{\alpha_i}(h) = c \text{ if and only if } i \in I\}$$

is called a $(n-m)$-dimensional face where c takes one value of $\{0, 1, \infty\}$ for each i. In particular, it is a type i reflection hypertorus if $I = \{i\}$ and $c = 1$ and a type i boundary divisor if $I = \{i\}$ and $c = 0$ or ∞. The union of all $(n-1)$-dimensional facets is called the set of subregular points. Then we analyze the local situation near a subregular point x. This will be used in proving the hyperbolic structure of H°.

Lemma 3.22. *For any $\kappa \in K$ the map $ev(\kappa)$ satisfies the following properties:*

(1) It maps locally biholomorphically into $\mathcal{F}_S^*(\kappa)$.
(2) Continuing $ev(\kappa)$ along a loop $\sigma \in \mathrm{Art}'(M)$ yields $\rho^*(\kappa, \sigma) ev(\kappa)$.

Proof. That evaluation map $ev(\kappa)$ is locally biholomorphic everywhere since (say $f_0 \neq 0$) the Wronskian

$$J = f_0^{n+1} \det(\partial_{\xi_i}(f_j/f_0))_{0 \le i,j \le n}$$

is precisely the Jacobian of the projective evaluation mapping in the affine chart $\{f_0 \neq 0\}$.

Statement 2 is clear since $\mathcal{F}_S^*(\kappa)$ is a fiber living on the base orbifold $W \backslash H^\circ$. □

Lemma 3.23. *We can pick local coordinates $y_1, y_2, \cdots, y_{n+1}$ and certain linear coordinates of $\mathcal{F}_S^*(\kappa)$ near x such that the evaluation map has the following form:*

$$ev(\kappa) = (y_1^{\frac{1}{2}-k_\alpha}, y_2, \cdots, y_{n+1}) \qquad \text{if } x \in H_\alpha^\circ$$
$$ev(\kappa) = (y_1^{1-k_p'}, \cdots, y_m^{1-k_p'}, y_{m+1}^{1-k_p''}, \cdots, y_{n+1}^{1-k_p''}) \qquad \text{if } x \in D_p^\circ.$$

Proof. From the computation in Section 3.2, we know that the eigenvalues of the residue map of the connection $\tilde{\nabla}$ along the mirror are $2k_\alpha$ and 0 with multiplicities 1 and n respectively while a half turn corresponds to a loop in $W \backslash H^\circ$.

Similarly, the eigenvalues of the residue map of the connection $\tilde{\nabla}$ along the boundary divisor are k_p' and k_p'', with multiplicities m and $n+1-m$ respectively say.

Then the evaluation map could be written as in the statement with respect to these coordinates. □

3.5. Hyperbolic complex ball

Then we finally arrive at the main result of this section. Inspired by the idea of Section 3.8 in Couwenberg [**5**], we can show the following fact.

Theorem 3.24. *For $\kappa \in K'_{\mathrm{hyp}}$, the image of the projective evaluation map*

$$\widehat{Pev} : \widehat{W\backslash H^\circ} \to \mathbb{P}^n(\mathbb{C})$$

is contained in the ball $\mathbb{B}^n(\mathbb{C})$.

Proof. Let $e_0, e_1, \cdots, e_n$ be the standard basis of $\mathcal{F}_S$, denote their dual sections in $\mathcal{F}_S^*$ by $e_0^*, e_1^*, \cdots, e_n^*$, then through the equivalence of monodromy representation and the reflection representation we can transfer the Hermitian structure in space $\mathcal{A}^{n+1}$ to the vector bundle $\mathcal{F}_S^*$ over the restricted real valued multiplicity function K' by defining $h^*(e_i^*, e_j^*) = h_{ij}^*$ as in (3.6). To prove this desired result, it suffices to show that

$$h^*(ev(\kappa, \cdot), ev(\kappa, \cdot)) > 0$$

on H° for hyperbolic κ, i.e., $\kappa \in K'_{\mathrm{hyp}}$.

Using the action of the Weyl group W on the adjoint torus H which corresponds to a complete Weyl Chamber decompostion Σ for the real vector space $P_{\mathbb{R}}^\vee = P^\vee \otimes \mathbb{R}$ spanned by the coweight lattice $P^\vee$, a smooth full compactification $H \to \hat{H}$ added by a boundary divisor with normal crossings could be realized. These boundary divisors are called the toric strata. Also we could compactify $\mathbb{C}^\times$ by adding two points $\{0, \infty\}$ which actually corresponds to a type A_1 Weyl chamber decomposition.

Let $\mathcal{D} = \cup_\alpha H_\alpha$ and let

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{C}^\times & \xrightarrow{\iota} & \hat{H} \times \mathbb{C}^\times \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle pr_1} \\ \mathbb{P}^1 & \xrightarrow{\iota'} & \hat{H} \end{array}$$

be a commutative diagram such that ι' is an embedding of a projective line in $\hat{H}$ which intersects every mirror and every irreducible toric divisor only in subregular points. In particular, the image $\iota'(\mathbb{C}^\times)$ is not contained in any reflection hypertorus or toric divisor. In fact, this desired projective line could be realized as follows: as $P^\vee \cong \mathbb{Z}^n$, any $p = \sum_i b_i p_i$ gives rise to a homomorphism $\gamma_p : \mathbb{C}^\times \to H$ which sends $\lambda \in \mathbb{C}^\times$ to $(\lambda^{b_1}, \lambda^{b_2}, \cdots, \lambda^{b_n}) \in (\mathbb{C}^\times)^n$. We now choose an element $p_1 = \sum_i b_i p_i$ with $b_1 = 1, b_i = 0$ for $i = 2, \cdots, n$, then we have a homomorphism

$$\begin{aligned} \gamma_1 : \mathbb{C}^\times &\to H \\ \lambda &\mapsto (\lambda, 1, \cdots, 1) \end{aligned}$$

If we compactify this 1-dimensional subtorus by adding two points $\{0, \infty\}$, this induced projective line intersects D_{p_1} and D_{-p_1} only in subregular points. Then we translate this projective line a little bit, i.e., multiply its coordinates by a complex number $1+\epsilon$ with ϵ very close to 0. By this way, we get a projective line which intersects mirrors and toric divisors only in subregular points.

In the diagram above, let ι map the second factor unchanged and pr_1 denote the projection map to the first factor. Let $a_1, \cdots, a_m$ be the points in $\mathbb{P}^1$ which are mapped by ι' into $\mathcal{D}$ and a_0, a_∞ be the points in $\mathbb{P}^1$ which are mapped by ι' into the toric divisor. Define a real valued function ϕ on $K' \times ((\mathbb{P}^1 \setminus \{a_0, a_1, \cdots, a_m, a_\infty\}) \times \mathbb{C}^\times)$ by:

$$\phi(\kappa, x) := h^*(ev(\kappa, \iota(x)), ev(\kappa, \iota(x))).$$

Here we write x instead of a point $(q, t) \in \mathbb{P}^1 \times \mathbb{C}^\times$.

Note that by monodromy invariance of h^* this defines a single valued continuous function. Then we conclude by the characterization in Lemma 3.23 that ϕ extends to a continuous function (also called ϕ) on $K' \times (\mathbb{P}^1 \times \mathbb{C}^\times)$.

We now investigate if this ϕ can take on negative values. If we denote the parabolic region by K_0, we observe that $\phi(\kappa, x) > 0$ for $\kappa \in K_0$. For parabolic κ, we always have the projection of x onto the $\mathbb{C}^\times$ part nonzero so that ϕ must be greater than 0. Define N by:

$$N := \{(\kappa, x) \in K' \times (\mathbb{P}^1 \times \mathbb{C}^\times) \mid \phi(\kappa, x) \leq 0\}.$$

Then N is closed by the continuity of the function ϕ. Because N is invariant under scalar multiplication in the second factor, we have that the projection N_K of N on K' along $\mathbb{P}^1 \times \mathbb{C}^\times$ is also closed.

Now suppose $\kappa \in \partial N_K$, then $\phi(\kappa, x) \geq 0$ otherwise κ cannot belong to the boundary of N_K by the continuity of ϕ. And we also have $\phi(\kappa, x_0) = 0$ for some $x_0 \in \mathbb{P}^1 \times \mathbb{C}^\times$. Suppose also that $\kappa \in K'_{\mathrm{hyp}}$. Because $ev(\kappa)$ is locally biholomorphic on $H^\circ \times \mathbb{C}^\times$ and the image $\iota'(\mathbb{C}^\times)$ of $\mathbb{C}^\times$ under ι' is not contained in any single irreducible component of the added divisor by a previous remark, we conclude that $\phi(\kappa, x) = 0$ implies that $x \in (a_0 \times \mathbb{C}^\times) \cup (a_1 \times \mathbb{C}^\times) \cup \cdots \cup (a_m \times \mathbb{C}^\times) \cup (a_\infty \times \mathbb{C}^\times)$ by the maximal principle. Hence $\phi(\kappa, \cdot)$ vanishes along some $\mathbb{C}^\times$-orbit.

Either $i \in \{1, \cdots, m\}$ or $\{0, \infty\}$, we know that at a non zero point x_0 in $a_i \times \mathbb{C}^\times$ we can write the evaluation map $ev(\kappa, x)$ locally of the form given in Lemma 3.23:

$$\begin{aligned} ev(\kappa, \iota(x)) &= (y_1^{\frac{1}{2}-k_\alpha}, y_2, \cdots, y_{n+1}) && \text{if } i \in \{1, \cdots, m\} \\ ev(\kappa, \iota(x)) &= (y_1^{1-k'_p}, \cdots, y_m^{1-k'_p}, y_{m+1}^{1-k''_p}, \cdots, y_{n+1}^{1-k''_p}) && \text{if } i \in \{0, \infty\}. \end{aligned}$$

Since κ lies in the hyperbolic region, so $\det(h^*)$ is of the signature $(1, n)$. While $\phi(\kappa, x) \geq 0$, then ϕ must possess the unique positive signature if we transform these above coordinates into the standard coordinates z_i for $1 \leq i \leq n+1$. Since the image $\iota(\mathbb{P}^1 \times \mathbb{C}^\times)$ is of dimension 2, we then have the following formula:

$$\phi(\kappa, x) = |z_i|^2 - |z_j|^2 \quad \text{for some } i, j$$

for x near x_0. Then we know that it must take value in an open interval containing 0 if x lies in a neighbourhood of x_0 which is in contradiction to that $\phi(\kappa, x) \geq 0$ for x takes value in a neighbourhood of x_0.

Therefore, we conclude that if $\kappa \in \partial N_K$ then κ is outside of K'_{hyp}. Since $K'_{\text{hyp}} \cup K_0$ is connected and not contained in N_K, we conclude that $K'_{\text{hyp}} \cup K_0$ is disjoint from N_K and hence K'_{hyp} is disjoint from N_K as well. This shows that $\phi(\kappa, x) > 0$ when $\kappa \in K'_{\text{hyp}}$. In particular we have that on the ι image of $\mathbb{C}^\times \times \mathbb{C}^\times$, evaluation maps into $\mathbb{B}^n \times \mathbb{C}^\times$, hence projective evaluation maps into $\mathbb{B}^n$. And the desired result follows by varying the map ι so that the images of ι' cover H°. $\square$

Remark 3.25. From previous computation, we can know that the monodromy along toric strata has only two different eigenvalues. But only those strata where one eigenvalue has multiplicity 1 and the other multiplicity n are mapped under the projective evaluation map to mirrors in the hyperbolic complex ball. And in fact, these strata act like complex reflections up to a scalar.

3.6. An example of ball quotients

In fact, under suitable so-called Schwarz conditions, we wish to find a ball quotient structure for our space $W\backslash H^\circ$ as follows

$$Pev : W\backslash H^\circ \to \Gamma\backslash\mathbb{B}$$

with Γ a discrete subgroup of $\mathrm{Aut}(\mathbb{B})$ with finite covolume, although we have not arrived there in this thesis. We even wish to find a modular interpretation for these (potential) ball quotients. Namely, does there exist such a commutative diagram

$$\begin{tikzcd} W\backslash H^\circ \arrow[r, "Pev"] \arrow[d] & \Gamma\backslash\mathbb{B} \arrow[d] \\ \mathcal{M} \arrow[r, "Per"] & \Gamma'\backslash\mathbb{B} \end{tikzcd}$$

that $\mathcal{M}$ is a suitable moduli space with Per a suitable period map. For R of type A_n with $k' = 0$, the answer is already given by the theory of Deligne-Mostow. In fact, the classical root system is just a special case of the theory of Deligne-Mostow with all the weights being equal. We also encounter the moduli space of Del Pezzo surfaces when we look at the type E_n. However, for the other root systems, we barely have any idea about them for the moment. We shall investigate the A_n case here since this example as well as type E_n cases strongly motivated current research presented in this thesis.

Example 3.26. For the root system R of type A_n, we have to impose a condition $k' = 0$. Let be given $n+3$ pairwise distinct points $z_0, \cdots, z_{n+2}$ on the projective line $\mathbb{P}^1$ and $n+3$ associated rational numbers $\mu_0, \cdots, \mu_{n+2} \in (0,1)$ with $\sum \mu_i = 2$. Fix $z_0 = 0$ and $z_{n+2} = \infty$, if we denote the simply connected torus by H', then H'° can be defined as

$$H'^{\circ} = \{(z_1, \cdots, z_{n+1}) \in (\mathbb{C}^{\times})^{n+1} \mid z_1 \cdots z_{n+1} = 1, z_i \neq z_j \\ \text{for each pair of distinct} (i,j)\}.$$

And then the adjoint torus

$$H^{\circ} = C_{n+1} \backslash H'^{\circ}$$

can be defined with $C_{n+1} = P^{\vee}/Q^{\vee}$ the cyclic group of order $n+1$. Let $\mathcal{M}_{0,n+3}$ denote the moduli space of genus 0 curve with $n+3$ marked points. Write $\mu_i = m_i/m$ with m being their smallest denominator, consider the algebraic curve $C(z)$ defined by the affine equation:

$$C(z) : y^m = \prod (\zeta - z_i)^{m_i}.$$

Then the periods of the cyclic cover of $\mathbb{C}$

$$\int_{z_i}^{z_{i+1}} \frac{d\zeta}{y}$$

are just the solutions of the hypergeometric equations. If we take $\mu_i = k$ for $i = 1, \cdots, n+1$ and the remaining $\mu_0 = \mu_{n+2} = (2-(n+1)k)/2$ so that it becomes our special hypergeometric system associated with the root system A_n. Let $\mathfrak{S}_\mu$ denote the subgroup of the symmetric group $\mathfrak{S}_{n+3}$ fixing $\mu = (\mu_0, \cdots, \mu_{n+2})$. The half integrality condition from the theory of Deligne-Mostow is given as follows:

$$\mu_i + \mu_j < 1 \Rightarrow (1 - \mu_i - \mu_j) \in \begin{cases} 1/\mathbb{N} & \text{if} \quad \mu_i \neq \mu_j \\ 2/\mathbb{N} & \text{if} \quad \mu_i = \mu_j \end{cases} \text{for all } i \neq j.$$

This happens to coincide with the Schwarz conditions for the special hypergeometric system with type A_n along the toric strata, along the mirrors and near the identity element:

$$\begin{aligned}(n-1)k/2 &= (1-\mu_0-\mu_1) \in 1/\mathbb{N}\\ (1-2k)/2 &= (1-\mu_1-\mu_{n+1})/2 \in 1/\mathbb{N}\\ ((n+1)k-1)/2 &= (1-\mu_0-\mu_{n+2})/2 \in 1/\mathbb{N}.\end{aligned}$$

If these conditions are satisfied, then we have a commutative diagram

$$\begin{tikzcd} W\backslash H^\circ \arrow[r, "Pev"] \arrow[d] & \Gamma_n\backslash\mathbb{B}^n \arrow[d] \\ \mathfrak{S}_\mu\backslash\mathcal{M}_{0,n+3} \arrow[r, "Per"] & \Gamma'_n\backslash\mathbb{B}^n \end{tikzcd}$$

with left vertical arrow a covering map and top arrow being an isomorphism onto a Heegner divisor complement.

CHAPTER 4

Frobenius structures

In this chapter, we present a quite preliminary result about the Frobenius structure on $H^\circ \times \mathbb{C}^\times$. Since the torsion free and flat connection $\tilde{\nabla}^\kappa$ defines an affine structure on $H^\circ \times \mathbb{C}^\times$, we would naturally speculate if there exists a Frobenius structure on it. It turns out to be the case in some weak sense, at least. We briefly introduce the basic definition of Frobenius manifolds in Section 4.1 and then construct a Froebnius algebra on our $H^\circ \times \mathbb{C}^\times$ in Section 4.2.

4.1. Frobenius manifolds

We shall in this section introduce the basic definition of Frobenius structures on a manifold briefly. For a good exposition, interested reader can consult the book by Manin [**26**] and the lecture notes of Looijenga [**25**].

For the moment, to us a $\mathbb{C}$-algebra is simply a $\mathbb{C}$-vector space E endowed with a $\mathbb{C}$-bilinear map (also referred as the product): $E \times E \to E,\ (u, v) \mapsto uv$ which is associative and a unit element $e \in E$ such that $e.u = 1.u = u$ for all $u \in E$. We often write 1 for e.

Definition 4.1. Let E be a $\mathbb{C}$-algebra which is commutative, associative and finite dimensional as a $\mathbb{C}$-vector space. A linear function on E, $F : E \to \mathbb{C}$ is called a *trace map* if the map $(u, v) \in E \times E \mapsto a(u, v) := F(uv)$ is a nondegenerate bilinear form. The pair (E, F) is called a *Frobenius algebra.*

Remark 4.2. The fact that the bilinear form a is nondegenerate is equivalent to that the resulting map $u \mapsto F(u.-)$ is a linear isomorphism of E onto its dual space E^* consisting of all the linear forms on E. We also need to point out that the trace map defined here is in general not the one that we usually associate a linear operator (if an element u of E is regarded as a linear operator $x \in E \mapsto ux \in E$) with a number.

Lemma 4.3. *The bilinear form a satisfies the associative law $a(uv, w) = a(u, vw)$. And conversely, any nondegenerate bilinear symmetric map $a : E \times E \to \mathbb{C}$ with the associative law determines a trace map on E.*

Proof. $a(uv,w) = F((uv)w) = F(u(vw)) = a(u,vw)$ since E is an associative $\mathbb{C}$-algebra, then the first statement follows.

Conversely, we can define a linear function I by $I(u) := a(u,e)$. Then we can define a new map $a' : E \times E \to \mathbb{C}$ as follows: $a'(u,v) := I(uv)$, but we have $I(uv) = a(uv,e) = a(u,v)$ by the associativity of a. This shows the newly defined map a' is the same as a which is also a nondegenerate bilinear symmetric form. The second statement follows. $\square$

Here are some simple examples of Frobenius algebra.

Example 4.4. (i) For the field $\mathbb{C}$ which could be viewed as a $\mathbb{C}$-algebra, we can define a trace map by a nonzero scalar multiplication $F : \mathbb{C} \to \mathbb{C}$; $u \mapsto \nu u$ for $\nu \neq 0$.

(ii) Let $E = \mathbb{C}[t]/(t^n)$ with $n \in \mathbb{Z}_+$. A linear form $F : E \to \mathbb{C}$ is a trace map if and only if $F(t^{n-1}) \neq 0$.

We can construct new Frobenius algebras out of old ones by the following ways: direct sums, tensor products and rescalings.

Direct sums. Let (E_1, F_1) and (E_2, F_2) be Frobenius $\mathbb{C}$-algebras. Then the vector space $E_1 \oplus E_2$ is an algebra for componentwise multiplication: $(u_1, v_1)(u_2, v_2) := (u_1u_2, v_1v_2)$. Its identity element is $(1, 1)$. Then $(u_1, u_2) \mapsto F_1(u_1) + F_2(u_2)$ is a trace map on $E_1 \oplus E_2$. It is easy to check that any trace map on $E_1 \oplus E_2$ must be of this form.

Tensor products. Let (E_1, F_1) and (E_2, F_2) be Frobenius algebras. Then the vector space $E_1 \otimes E_2$ is an algebra whose product can be defined as $(\sum_i u_i \otimes v_i)(\sum_j u'_j \otimes v'_j) = \sum_{i,j} u_iu'_j \otimes v_iv'_j)$. Its identity element is $1 \otimes 1$. Then $u \otimes v \mapsto F_1(u)F_2(v)$ is a trace map on $E_1 \otimes E_2$.

Rescaling. Let E be a Frobenius algebra. For any nonzero scalar $\nu \in \mathbb{C}$ we can define a new algebra structure with the same underlying vector space, i.e., a new product given as $u * v = \nu uv$. Its identity element is $\nu^{-1}e$. Then νF is a trace map on the new algebra $(E, *)$.

Now let us see what kind of role the associativity condition plays here? If we are given a Frobenius algebra, we can also consider the trilinear map $T : E \times E \times E \to \mathbb{C}$ defined by $T(u,v,w) := F(uvw)$. But conversely, if we are only given a vector space E, a trilinear map $T : E \times E \times E \to \mathbb{C}$, and an element $e \in E$, does T defines a Frobenius algebra structure on E? The answer is obviously no. We must impose some additional conditions so that T can be used to define a Frobenius algebra on E. First T must be required to be symmetric. And we also want that the bilinear form $(u,v) \in E \times E \mapsto T(u,v,e) \in \mathbb{C}$ is nondegenerate. We thus have defined a bilinear map (product) $E \times E \to E$, $(u,v) \mapsto uv$ characterized by that $T(uv,x,1) = T(u,v,x)$ for all $x \in E$. Since T is symmetric, the product is commutative and e becomes the identity element of E for ue is characterized by $T(ue,e,x) =$

$T(u, e, x)$ for all $x \in E$. Besides these two conditions, the associativity dose not hold a priori and thus has to be endowed. This means we want that $T(uv, w, x) = T(u, vw, x)$ for all $u, v, w, x \in E$. In fact, we can write out this condition in terms of a basis of E. If $\{u_1, \cdots, u_n\}$ is a basis of E, define $T_{ijk} := T(u_i, u_j, u_k)$, then $(a_{jk} := T_{1jk})_{jk}$ is a nondegenerate matrix. Let $(a^{jk})_{jk}$ denote its inverse matrix, then we have

$$u_i u_j = T_{ijk} a^{kl} u_l$$

where the Einstein summation convention is used. The above associativity condition also means we want that $T(u_i u_j, u_k, u_l) = T(u_i, u_j u_k, u_l)$ in terms of the basis. So we have $T(T_{ijp} a^{pq} u_q, u_k, u_l) = T(u_i, T_{jkp} a^{pq} u_q, u_l)$ which is equivalent to

$$T_{ijp} a^{pq} T_{qkl} = T_{jkp} a^{pq} T_{iql}. \tag{Ass.}$$

This is a system of equations which must be satisfied in order that the product being associative.

Now let be given a complex manifold M whose holomorphic tangent bundle is denoted by TM. We are also given on TM a nondegenerate symmetric bilinear form a and a symmetric trilinear form T, both depending on holomorphically on the base point. The product of this bundle can be characterized by the property that $a(XY, Z) = T(X, Y, Z)$, denoted by $\cdot : TM \times TM \to \mathbb{C}; X \cdot Y \mapsto XY$. It is clear that this product is commutative by the symmetry of T. We use ∇ to denote the complex counterpart *Levi-Civita* connection on the holomorphic tangent bundle TM which is characterized by the following 2 properties.

compatibility: $Z(a(X, Y)) = a(\nabla_Z X, Y) + a(X, \nabla_Z Y)$,
torsion freeness: $\nabla_X Y - \nabla_Y X = [X, Y]$

and its curvature form is given by

$$\mathrm{R}(\nabla)(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

We can then define a one-parameter family of connections $\nabla(\mu)$ on this bundle by

$$\nabla(\mu)_X Y := \nabla_X Y + \mu X \cdot Y, \quad \mu \in \mathbb{C}.$$

By the commutativity of the product we immediately have

$$\nabla(\mu)_X Y - \nabla(\mu)_Y X - [X, Y] = \nabla_X Y - \nabla_Y X - [X, Y] = 0$$

which shows that $\nabla(\mu)$ is torsion free. If for a local vector field X on M, ι_X denotes the multiplication operator on vector fields: $\iota_X(Y) := X \cdot Y$, then we can define a new tensor

$$\mathrm{R}'(\nabla)(X, Y) := [\nabla_X, \iota_Y] - [\nabla_Y, \iota_X] - \iota_{[X,Y]}$$

which is a holomorphic 2-form taking values in the symmetric endomorphism of TM. It's clear that $a(\mathrm{R}'(\nabla)(X,Y)Z,W)$ is antisymmetric in (X,Y) and symmetric in (Z,W).

Proposition 4.5. *The following statements are equivalent:*

(i) ∇ *is flat, the product is associative and the trilinear form* $T(X,Y,Z)$ *locally is given by* $T(X,Y,Z) = \nabla_X\nabla_Y\nabla_Z\Phi$ *where* $\Phi : U \to \mathbb{C}$ *is a holomorphic function on a domain* $U \subset M$*.*

(ii) ∇ *is flat, the product is associative and* $\mathrm{R}' \equiv 0$*.*

(iii) The connection $\nabla(\mu)$ *is flat for any* $\mu \in \mathbb{C}$*.*

Proof. First prove $(ii) \Leftrightarrow (iii)$. We have

$$\begin{aligned}\nabla(\mu)_X\nabla(\mu)_Y &= (\nabla_X + \mu\iota_X)(\nabla_Y + \mu\iota_Y)\\ &= \nabla_X\nabla_Y + \mu(\iota_X\nabla_Y + \nabla_X\iota_Y) + \mu^2\iota_X\iota_Y.\end{aligned}$$

Similarly,

$$\begin{aligned}\nabla(\mu)_Y\nabla(\mu)_X &= \nabla_Y\nabla_X + \mu(\iota_Y\nabla_X + \nabla_Y\iota_X) + \mu^2\iota_Y\iota_X,\\ \nabla(\mu)_{[X,Y]} &= \nabla_{[X,Y]} + \mu\iota_{[X,Y]}.\end{aligned}$$

Then we have

$$\begin{aligned}\mathrm{R}(\nabla(\mu))(X,Y) &= \nabla(\mu)_X\nabla(\mu)_Y - \nabla(\mu)_Y\nabla(\mu)_X - \nabla(\mu)_{[X,Y]}\\ &= \mathrm{R}(\nabla)(X,Y) + \mu\mathrm{R}'(\nabla)(X,Y) + \mu^2(\iota_X\iota_Y - \iota_Y\iota_X).\end{aligned}$$

So if ∇ is flat, i.e., $\mathrm{R}(\nabla) = 0$. We then see that $\nabla(\mu)$ is flat for all μ if and only if $\mathrm{R}'(\nabla) = 0$ and $\iota_X\iota_Y = \iota_Y\iota_X$ for all X, Y. While the condition that $\iota_X\iota_Y = \iota_Y\iota_X$ for all X, Y is equivalent to that $X\cdot(Y\cdot Z) = Y\cdot(X\cdot Z)$ for all X, Y, Z. Since the left hand side $X\cdot(Y\cdot Z) = X\cdot(Z\cdot Y)$ and the right hand side $Y\cdot(X\cdot Z) = (X\cdot Z)\cdot Y$ by the commutativity of the product. This is just the associativity property. So $(ii) \Leftrightarrow (iii)$ follows.

Now let us prove $(i) \Leftrightarrow (ii)$. Since a is flat we can pass all the things to a flat chart (U,φ) such that $D = \varphi(U) \subset \mathbb{C}^n$ is an open polydisk. Under this setting, a has constant coefficients, ∇ becomes the usual derivation and the flat vector fields are just the constant ones. Suppose we are given holomorphic functions $f_{ijk} : D \to \mathbb{C}$ for $1 \le i,j,k \le n$. It is well-known that these can arise as the third order partial derivatives of a holomorphic function Φ if and only if $\partial_l f_{ijk}$ is symmetric in all its indices. In other words, if f is a trilinear form on the tangent bundle of D, then there exists a holomorphic function Φ such that $f(X,Y,Z) = \nabla_X\nabla_Y\nabla_Z\Phi$ for all triples of flat vector fields (X,Y,Z) if and only if $X(f(Y,Z,W))$ is symmetric in its all arguments for all quadruples of flat vector fields (X,Y,Z,W). Since we have $f(X,Y,Z) = a(XY,Z)$ and we already know that f is symmetric in its three arguments. We have

$$\begin{aligned} Xa(Y\cdot Z, W) &= a(\nabla_X(Y\cdot Z), W) + a(Y\cdot Z, \nabla_X W) \\ &= a(\nabla_X(Y\cdot Z), W) \end{aligned}$$

But since X, Y, Z are all flat, we also have

$$\mathrm{R}'(\nabla)(X,Y)Z = \nabla_X(Y\cdot Z) - \nabla_Y(X\cdot Z)$$

and then it is clear that $Xa(Y\cdot Z, W)$ is symmetric in X and Y if and only if $\mathrm{R}' = 0$. □

Remark 4.6. The function Φ that appears in Statement (i) of Proposition 4.5 is called a (local) *potential function*. Since here only its third order derivatives matter, it is (in terms of flat coordinates $(z^1, \cdots, z^n)$) unique up to a polynomial of degree two. In particular, a potential function needs not be defined on all of M. The associativity equation (Ass.) now is read as a highly nontrivial system of partial differential equations: if $(z^1, \cdots, z^n)$ is a system of flat coordinates and $\partial_\nu := \frac{\partial}{\partial z^\nu}$, then we require that for all i, j, k, l,

$$(\partial_i\partial_j\partial_p\Phi)a^{pq}(\partial_q\partial_k\partial_l\Phi) = (\partial_j\partial_k\partial_p\Phi)a^{pq}(\partial_i\partial_q\partial_l\Phi) \qquad \text{(WDVV)}$$

These are known as the *Witten-Dijkgraaf-Verlinde-Verlinde* equations.

Then we are properly prepared to introduce the main notion of this chapter.

Definition 4.7. A complex manifold M is called a *Frobenius manifold* if its holomorphic tangent bundle is fiberwisely endowed with the structure of a Frobenius algebra $(\cdot, F, e)$ satisfying

(i) the equivalent conditions of Proposition 4.5 are fulfilled for the associated symmetric bilinear and trilinear forms a and T, and

(ii) the identity field e on M is flat for the Levi-Civita connection of a.

Here are some examples of Frobenius manifolds.

Example 4.8. (i) The trivial example is $M = \mathbb{C}^n$ whose coordinates are $(z^1, \cdots, z^n)$, $a = \sum_i (dz^i)^2$ and product $\partial_i \cdot \partial_i = \partial_i$. A potential function is a cubic form $\Phi(z) = \frac{1}{6}\sum_i (z^i)^3$ and the family of connections is given by $\nabla(\mu)_{\partial_i}\partial_j = \mu\delta^i_j\partial_i$.

(ii) (Two-dimensional case) In this case the product on a vector space E of dimension two with nonzero unit e is automatically associative. We then have E is isomorphic to the semisimple $\mathbb{C}\oplus\mathbb{C}$ or to the nonsemisimple $\mathbb{C}[y]/(y^2)$. It remains to find the potential functions. Let e be the unit vector field and F the trace differential. Since e is flat, $a(e,e) = F(e\cdot e)$ is constant, say equal to $c \in \mathbb{C}$. There are two cases depending on whether c is 0 or not.

We first do the case $c = 0$. Then we can find flat coordinates (z, w) such that $e = \partial_z$ and $a = dz \otimes dw + dw \otimes dz$. Since we have $a(\partial_z \cdot \partial_z, \partial_z) = a(\partial_z, \partial_z) = 0$ and $a(\partial_z \cdot \partial_z, \partial_w) = a(\partial_z, \partial_w) = 1$, it follows that $\Phi_{zzz} = 0$ and $\Phi_{zzw} = 1$. But since $\partial_z \cdot \partial_w = \partial_w$, we must also have $\Phi_{zwz} = 1$ and $\Phi_{zww} = 0$. It follows that $\Phi(z, w) = \frac{1}{2} z^2 w + f(w)$ up to quadratic terms, where f is holomorphic.

If $c \neq 0$, then we can find flat coordinates (z, w) such that $e = \partial_z$ and $a = c dz \otimes dz + c dw \otimes dw$. Then we want that $\Phi_{zzz} = c$, $\Phi_{zzw} = 0$, $\Phi_{zwz} = 0$, $\Phi_{zww} = c$. It follows that $\Phi(z, w) = \frac{1}{6} c z^3 + \frac{1}{2} c z w^2 + f(w)$ up to quadratic terms, where f is holomorphic.

Conversely, in both cases, with these choice of e and a, any Φ of the form defines a Frobenius manifold.

Remark 4.9. The most important class of examples is furnished by quantum cohomology which in fact motivated the definition in the first place. And another beautiful class of examples is furnished by the space of polynomials which is due to Saito and Dubrovin. But we shall not elaborate these two important classes of Frobenius manifolds over here. Interested readers can consult Manin [**26**] for detailed explanation.

4.2. Frobenius algebras on $H^\circ \times \mathbb{C}^\times$

We can already see from preceding chapters that there exists a family of flat connections $\tilde{\nabla}^\kappa$ on $H^\circ \times \mathbb{C}^\times$ with explicit formula. We can hence accordingly define a product structure for each $\kappa \in K$ on the tangent bundle of $H^\circ \times \mathbb{C}^\times$ which happens to give rise to a Frobenius algebra on it.

First we already have a family of connections ∇^κ on H° such that

$$\begin{cases} \nabla^\kappa(\zeta) = (\nabla^0 + \Omega^\kappa)\zeta \\ \nabla^\kappa \nabla^\kappa(\zeta) = -\zeta \wedge A^\kappa, \end{cases}$$

among which

$$\Omega^\kappa := \frac{1}{2} \sum_{\alpha > 0} k_\alpha \frac{e^\alpha + 1}{e^\alpha - 1} d\alpha \otimes d\alpha \otimes \partial_{\alpha^\vee} + (B^\kappa)^*.$$

Then we accordingly define $\tilde{\Omega}^\kappa$ on $H^\circ \times \mathbb{C}^\times$ by

$$\tilde{\Omega}^\kappa : \begin{cases} \zeta \in \Omega_{H^\circ} \mapsto \dfrac{1}{2} \displaystyle\sum_{\alpha > 0} k_\alpha \zeta(X_\alpha) d\alpha \otimes d\alpha + (B^\kappa)^*(\zeta) - \zeta \otimes \dfrac{dt}{t} - \dfrac{dt}{t} \otimes \zeta, \\ \dfrac{dt}{t} \mapsto A^\kappa - \dfrac{dt}{t} \otimes \dfrac{dt}{t}. \end{cases}$$

to get a family of flat connections on $H^\circ \times \mathbb{C}^\times$. Based on this, we can write $\tilde{\Omega}^\kappa$ explicitly:

$$\begin{aligned}\tilde{\Omega}^\kappa :=&\frac{1}{2}\sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1} d\alpha\otimes d\alpha\otimes\partial_{\alpha^\vee} + (B^\kappa)^* + c^\kappa\sum_{\alpha>0} d\alpha\otimes d\alpha\otimes t\frac{\partial}{\partial t}\\ &-\sum_{\alpha_i\in\mathfrak{B}} d\alpha_i\otimes\frac{dt}{t}\otimes\partial_{p_i} - \sum_{\alpha_i\in\mathfrak{B}}\frac{dt}{t}\otimes d\alpha_i\otimes\partial_{p_i} - \frac{dt}{t}\otimes\frac{dt}{t}\otimes t\frac{\partial}{\partial t}.\end{aligned}$$

We then transfer the connections $\tilde{\nabla}^\kappa$ defined on the cotangent bundle of $H^\circ \times \mathbb{C}^\times$ to the connections defined on the tangent bundle of $H^\circ \times \mathbb{C}^\times$:

$$\begin{aligned}(\tilde{\Omega}^\kappa)^* :=&\frac{1}{2}\sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1} d\alpha\otimes\partial_{\alpha^\vee}\otimes d\alpha + ((B^\kappa)^*)' + c^\kappa\sum_{\alpha>0} d\alpha\otimes t\frac{\partial}{\partial t}\otimes d\alpha\\ &-\sum_{\alpha_i\in\mathfrak{B}} d\alpha_i\otimes\partial_{p_i}\otimes\frac{dt}{t} - \sum_{\alpha_i\in\mathfrak{B}}\frac{dt}{t}\otimes\partial_{p_i}\otimes d\alpha_i - \frac{dt}{t}\otimes t\frac{\partial}{\partial t}\otimes\frac{dt}{t}.\end{aligned}$$

Since $T_{(p,t)}(H^\circ \times \mathbb{C}^\times) = T_pH^\circ \oplus T_t\mathbb{C}^\times$, we can write a vector field $\tilde{X}$ on $H^\circ \times \mathbb{C}^\times$ in the following form:

$$\tilde{X} = X(p,t) + \lambda_1(p,t)t\frac{\partial}{\partial t},$$

among which $X(p,t)$ is a vector field on H° and $\lambda_1(p,t)$ is a holomorphic function depending on both p and t. Here we write $\tilde{X} = X + \lambda_1 t\frac{\partial}{\partial t}$ just for convenience.

Let $\tilde{Y} = Y + \lambda_2 t\frac{\partial}{\partial t}$, inspired by the flat connection $\tilde{\nabla}^\kappa$, we define a product for each κ on each fiber of the tangent bundle of $H^\circ \times \mathbb{C}^\times$ by

$$\begin{aligned}\tilde{X}\cdot_\kappa\tilde{Y}|_{(p,t)} :=&\frac{1}{2}\sum_{\alpha>0} k_\alpha\frac{e^\alpha+1}{e^\alpha-1}\alpha(X)\alpha(Y)\alpha^\vee + b^\kappa(X,Y) + c^\kappa\sum_{\alpha>0}\alpha(X)\alpha(Y)t\frac{\partial}{\partial t}\\ &-\sum_{\alpha_i\in\mathfrak{B}}\alpha_i(X)\lambda_2 p_i - \sum_{\alpha_i\in\mathfrak{B}}\lambda_1 p_i\alpha_i(Y) - \lambda_1\lambda_2 t\frac{\partial}{\partial t}\\ =&\frac{1}{2}\sum_{\alpha>0} k_\alpha\frac{e^\alpha+1}{e^\alpha-1}\alpha(X)\alpha(Y)\alpha^\vee + b^\kappa(X,Y) + a^\kappa(X,Y)t\frac{\partial}{\partial t}\\ &-\lambda_2 X - \lambda_1 Y - \lambda_1\lambda_2 t\frac{\partial}{\partial t}.\end{aligned} \tag{4.1}$$

We already know that a^κ is a symmetric bilinear form on $\mathfrak{h}$:

$$a^\kappa : \mathfrak{h}\times\mathfrak{h}\to\mathbb{C}.$$

We can extend a^κ to be a symmetric bilinear form on $\mathfrak{h} \oplus \mathbb{C}$ which is the tangent space of $H^\circ \times \mathbb{C}^\times$ at (p,t) by defining

$$\begin{cases} a^\kappa(X, t\dfrac{\partial}{\partial t}) = 0 \\ a^\kappa(t\dfrac{\partial}{\partial t}, t\dfrac{\partial}{\partial t}) = -1. \end{cases}$$

Now is $a^\kappa(\alpha^\vee, \cdot)$ a linear form whose zero set is the hyperplane which is perpendicular to α and therefore it is proportional to α. By evaluating both sides on $\alpha^\vee$ we see that

$$a^\kappa(\alpha^\vee, \cdot) = \frac{a^\kappa(\alpha^\vee, \alpha^\vee)}{\alpha(\alpha^\vee)}\alpha.$$

Remark 4.10. We also notice that

$$(t\frac{\partial}{\partial t}) \cdot_\kappa \tilde{Y} = -Y - \lambda_2 t\frac{\partial}{\partial t} = -\tilde{Y},$$

from which we can see that $-t\frac{\partial}{\partial t}$ plays a role of identity in this algebra.

Theorem 4.11. *The product structure $\cdot_\kappa$ defined on $T(H^\circ \times \mathbb{C}^\times)$ by (4.1) makes each fiber of $T(H^\circ \times \mathbb{C}^\times)$ into a Frobenius algebra.*

Proof. In order to see this product structure indeed defines a Frobenius algebra on each fiber of the tangent bundle of $H^\circ \times \mathbb{C}^\times$, we need to verify 3 properties:

1. the product is commutative,
2. the product satisfies the associativity law with respect to the symmetric bilinear form a^κ (sometimes also called Frobenius condition), with this property the trace map can be determined by Lemma 4.3,
3. the product is associative.

1. commutativity of the product.

It's quite obvious since the expression for $\tilde{X} \cdot_\kappa \tilde{Y}$ is symmetric in $\{\tilde{X}, \tilde{Y}\}$.

2. Frobenius condition.

Let $\tilde{Z} = Z + \lambda_3 t\frac{\partial}{\partial t}$, then we have

$$
\begin{aligned}
& a^\kappa(\tilde{X} \cdot_\kappa \tilde{Y}, \tilde{Z}) \\
=& \frac{1}{2}\sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1}\alpha(X)\alpha(Y)a^\kappa(\alpha^\vee, \tilde{Z}) + a^\kappa(b^\kappa(X,Y),Z) + a^\kappa(X,Y)a^\kappa(t\frac{\partial}{\partial t}, \tilde{Z}) \\
& - \lambda_2 a^\kappa(X,\tilde{Z}) - \lambda_1 a^\kappa(Y,\tilde{Z}) - \lambda_1\lambda_2 a^\kappa(t\frac{\partial}{\partial t}, \tilde{Z}) \\
=& \frac{1}{2}\sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1}\cdot\frac{a^\kappa(\alpha^\vee,\alpha^\vee)}{\alpha(\alpha^\vee)}\alpha(X)\alpha(Y)\alpha(Z) + a^\kappa(b^\kappa(X,Y),Z) \\
& - \lambda_3 a^\kappa(X,Y) - \lambda_2 a^\kappa(X,Z) - \lambda_1 a^\kappa(Y,Z) + \lambda_1\lambda_2\lambda_3.
\end{aligned}
$$

From this, we can see that

$$a^\kappa(\tilde{X} \cdot_\kappa \tilde{Y}, \tilde{Z}) = a^\kappa(\tilde{X}, \tilde{Y} \cdot_\kappa \tilde{Z}),$$

since this expression is fully symmetric in $\{\tilde{X}, \tilde{Y}, \tilde{Z}\}$. In fact, the symmetry of $a^\kappa(b^\kappa(X,Y),Z)$ is guaranteed by Condition (3) of Lemma 2.10.

3. associativity of the product.

Let us look at the connection $\tilde{\nabla}^\kappa(\mu)$ defined by

$$\tilde{\nabla}^\kappa(\mu)_{\tilde{X}}\tilde{Y} := \tilde{\nabla}^\kappa_{\tilde{X}}\tilde{Y} + \mu\tilde{X} \cdot_\kappa \tilde{Y}, \quad \mu \neq -1.$$

Written out,

$$
\begin{aligned}
\tilde{\nabla}^\kappa(\mu)_{\tilde{X}}\tilde{Y} =& \frac{1}{2}(1+\mu)\sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1}\alpha(X)\alpha(Y)\alpha^\vee + \frac{1}{2}(1+\mu)k'\sum_{\alpha>0}\alpha(X)\alpha(Y)\alpha' \\
&+ (1+\mu)c^\kappa\sum_{\alpha>0}\alpha(X)\alpha(Y)t\frac{\partial}{\partial t} - (1+\mu)\lambda_2 X - (1+\mu)\lambda_1 Y - (1+\mu)\lambda_1\lambda_2 t\frac{\partial}{\partial t}.
\end{aligned}
$$

If we replace a tangent vector $(1+\mu)v \in \mathfrak{h}\oplus\mathbb{C}$ by v_{new}, then by Remark 4.12 the above connection becomes a connection on the newly defined tangent space $\mathfrak{h}\oplus\mathbb{C}$ with the connection form $(\tilde{\Omega}^\kappa)^*$. But we already know that $\tilde{\nabla}^\kappa$ is flat, so we can see that $\tilde{\nabla}^\kappa(\mu)$ ($\mu \in \mathbb{C}$ and $\mu \neq -1$) is also flat. Therefore, the associativity of the product follows by Proposition 4.5. □

Remark 4.12. In fact, there are two ways to look at the dual pairing $\mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$; $(\alpha, X) \mapsto \alpha(X)$ when we rescale the tangent space $\mathfrak{h}\oplus\mathbb{C}$. One way is to rescale the $\mathfrak{h}^*$ at the same time by defining $\alpha_{\text{new}} := (1+\mu)^{-1}\alpha$ so that $\alpha_{\text{new}}(X_{\text{new}}) = \alpha(X)$. Another way is to rescale the dual pairing by $(\alpha, X_{\text{new}})_{\text{new}} := \alpha(X)$.

Remark 4.13. In fact, our Frobenius algebra given above includes the Frobenius algebra constructed by Bryan and Gholampour in [**4**] as a special case, which requires $k' = 0$ for type A_n and $k = k'$ for type $BCFG$. They provided a proof for the associativity of the product from a point of view of Gromov-Witten theory.

Corollary 4.14. *The Weyl group acts on the tangent bundle by automorphisms. Namely, if we define*

$$g(e^\alpha) = e^{g(\alpha)}$$

for $g \in W$, then for $\tilde{X}, \tilde{Y} \in \Gamma(T(H^\circ \times \mathbb{C}^\times))$, we have

$$g(\tilde{X} \cdot_\kappa \tilde{Y}) = g(\tilde{X}) \cdot_\kappa g(\tilde{Y}).$$

PROOF. Let s_β be the reflection about the hyperplane orthogonal to β. By [**2**], s_β permute the positive roots other than β. And since the terms

$$\frac{e^\alpha + 1}{e^\alpha - 1}\partial_{\alpha^\vee} \quad \text{and} \quad \alpha(X)\alpha(Y)$$

remain unchanged under $\alpha \to -\alpha$, the effect of s_β to the formula for $\tilde{X} \cdot_\kappa \tilde{Y}$ is to permute the order of the sum:

$$\begin{aligned}
&s_\beta(\tilde{X} \cdot_\kappa \tilde{Y})\\
=&\frac{1}{2}\sum_{\alpha>0} k_{s_\beta(\alpha)} \frac{e^{s_\beta(\alpha)}+1}{e^{s_\beta(\alpha)}-1} s_\beta(\alpha)(s_\beta X) s_\beta(\alpha)(s_\beta Y)\partial_{s_\beta(\alpha^\vee)} + b^\kappa(s_\beta X, s_\beta Y)\\
&+ a^\kappa(s_\beta X, s_\beta Y) s_\beta(t\frac{\partial}{\partial t}) - s_\beta(\lambda_2 X) - s_\beta(\lambda_1 Y) - s_\beta(\lambda_1\lambda_2 t\frac{\partial}{\partial t})\\
=&\frac{1}{2}\sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1}\alpha(s_\beta X)\alpha(s_\beta Y)\partial_{\alpha^\vee} + b^\kappa(s_\beta X, s_\beta Y)\\
&+ a^\kappa(s_\beta X, s_\beta Y) t\frac{\partial}{\partial t} - \lambda_2 s_\beta X - \lambda_1 s_\beta Y - \lambda_1\lambda_2 t\frac{\partial}{\partial t}\\
=&(s_\beta X + \lambda_1 t\frac{\partial}{\partial t}) \cdot_\kappa (s_\beta Y + \lambda_2 t\frac{\partial}{\partial t})\\
=&(s_\beta X + s_\beta(\lambda_1 t\frac{\partial}{\partial t})) \cdot_\kappa (s_\beta Y + s_\beta(\lambda_2 t\frac{\partial}{\partial t}))\\
=&s_\beta(\tilde{X}) \cdot_\kappa s_\beta(\tilde{Y})
\end{aligned}$$

since $s_\beta(\lambda_i t\frac{\partial}{\partial t}) = \lambda_i t\frac{\partial}{\partial t}$. Then the corollary follows. □

Therefore, we construct a W-invariant Frobenius algebra on $H^\circ \times \mathbb{C}^\times$.

APPENDIX A

Classification of (extended) Dynkin diagrams

This appendix is to address the question which arises in Chapter 3 when dealing with the hyperbolic structure on algebraic torus. In order to determine the hyperbolic region for the projective structure defined on the algebraic torus, we have to compute the determinant of the Gram matrix of the invariant Hermitian form induced on it. The basic technique used there comes from the classification theory of the (extended) Dynkin (Coxeter) diagrams. Here we shall briefly explain this theory and then we are able to prove Theorem 3.18 with the help of this theory. There are several books which have a good exposition on this theory Bourbaki [**2**], Humphreys [**18**], as well as the lecture notes of Heckman [**13**].

A.1. Root systems

Let V be a finite dimensional Euclidean vector space with an inner product $(\cdot,\cdot)$. For a nonzero vector $\alpha \in V$, there corresponds an orthogonal reflection s_α with the hyperplane perpendicular to α being the mirror. This reflection could be written as

$$s_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$$

for any $\beta \in V$. We can easily check that

$$s_\alpha(\alpha) = -\alpha \text{ and } s_\alpha(\beta) = \beta \text{ for } (\beta,\alpha) = 0.$$

Then $s_\alpha^2 = 1$ follows from the above formula directly. We recall the definition of a *root system* first.

A finite subset R of V is called a *root system* if it does not contain 0 and generates V such that any s_α leaves R stable and $s_\alpha(\beta) \in \beta + \mathbb{Z}\alpha$ for any $\alpha, \beta \in R$. Any vector belonging to R is called a root. The dimension of V is called the *rank* of the system. The group $W(R)$ generated by the s_α is called the *Weyl group* of R.

This root system R is said to be *reduced* if $R \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for any $\alpha \in R$, and said to be *irreducible* if nonempty R can not be decomposed as a direct sum of two nonempty root systems.

We denote the complement of all mirrors by V° and a connected component of V° is called a *Weyl chamber*. Once we fix a Weyl chamber and denote it by C, it gives rise to a corresponding partition

$$R = R_+ \sqcup R_-$$

of R into positive and negative parts. This Weyl chamber is called a *positive* Weyl chamber. In fact, the positive part R_+ is given by

$$R_+ = \{\alpha \in R \mid (\alpha, \gamma) > 0 \text{ for } \forall \gamma \in C\}.$$

Conversely, once a positive roots R_+ is fixed, a corresponding Weyl chamber is also determined by

$$C = \{\gamma \in V \mid (\gamma, \alpha) > 0 \text{ for } \forall \alpha \in R_+\}.$$

We can see from above that C and R_+ mutually determines each other. A root of R_+ is called *simple* if it can not be written as a sum of two positive roots of R_+. The set of all the simple roots is called a *fundamental system* of R, denoted by $\mathfrak{B}$. The simple roots in R_+ are linearly independent and hence become a basis of V. The highest root $\tilde{\alpha}$ of R is defined as $\tilde{\alpha} = \sum_{i=1}^n n_i \alpha_i$ such that $n_i \geq p_i$ $(i = 1, \cdots, n)$ for any root $\alpha = \sum_{1=1}^n p_i \alpha_i$.

We write the so-called *Cartan integers* as follows

$$n_{\alpha\beta} = 2\frac{(\alpha, \beta)}{(\beta, \beta)} = -2\frac{\|\alpha\|}{\|\beta\|} \cos \frac{\pi}{m_{\alpha\beta}}$$

for $\alpha, \beta \in R$, where $m_{\alpha\beta}$ denotes the order of $s_\alpha s_\beta$.

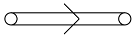
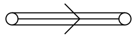
If we write n_{ij} instead of $n_{\alpha_i \alpha_j}$ for $\alpha_i, \alpha_j \in \mathfrak{B}$, we have all the possibilities of n_{ij} and m_{ij} for a root system in the following lemma.

Lemma A.1. *There are only finitely many possibilities for Cartan integers n_{ij} and the order m_{ij} of $s_\alpha s_\beta$ for a root system R, up to interchanging i and j:*

1) $n_{ii} = 0$; $m_{ii} = 1$;
2) $n_{ij} = n_{ji} = 0$; $m_{ij} = 2$;
3) $n_{ij} = n_{ji} = -1$; $m_{ij} = 3$;
4) $n_{ij} = -2$, $n_{ji} = -1$; $m_{ij} = 4$;
5) $n_{ij} = -3$, $n_{ji} = -1$; $m_{ij} = 6$.

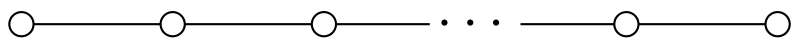
The matrix $N = (n_{ij})$ and $M = (m_{ij})$ are called the *Cartan matrix* and *Coxeter matrix* respectively for R. Then we can introduce the *Dynkin diagram* for a root system R as follows.

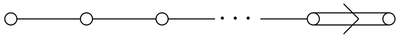
The *Dynkin diagram* of a root system R associated with its Coxeter matrix M is a marked graph with n nodes labelled $1, \cdots, n$ and nodes are joined in the following way: two distinct nodes are joined by 0,1,2 or 3 bonds in case 2), 3), 4) and 5) above in the Lemma A.1. Moreover, in case 4) and 5), that is when $|n_{ij}| > 1$, an inequality sign $>$ is placed on the double or triple bond joining the nodes corresponding to i and j oriented towards the node corresponding to j:

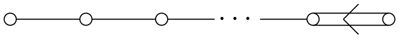
 (for $n_{ij} = -2$)  (for $n_{ij} = -3$)

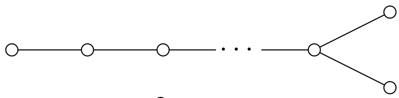
Then we have the list of possible Dynkin diagrams.

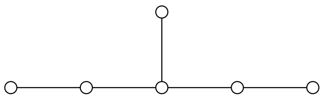
Theorem A.2. *Suppose R is an irreducible reduced root system, its Dynkin diagram is isomorphic to one of the following types of diagrams:*

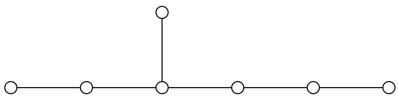
A_n 

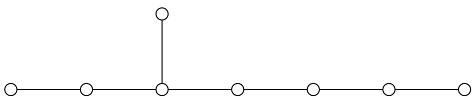
B_n 

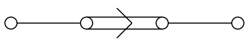
C_n 

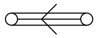
D_n 

E_6 

E_7 

E_8 

F_4 

G_2 

No two of these diagrams are isomorphic.

Once we have a Coxeter matrix $M = (m_{ij})$ for a root system R, we can define the corresponding *Gram matrix* $G(M) = (g_{ij})$ in this way

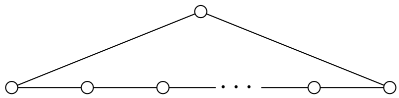
$$G(M) := (g_{ij} = -2\cos(\pi/m_{ij}))$$

which could be regarded as the metric of the inner product (up to a scalar) with respect to these simple roots.

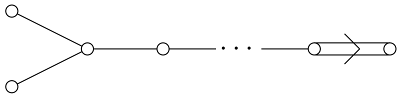
Now in order to take affine-linear transformation of affine space V into consideration, we can introduce *affine root system* for V by adding a new root α_0 to the fundamental system which is defined as $\alpha_0 = -\tilde{\alpha}$. Then the corresponding *extended Dynkin diagram* is defined as follows: add a node corresponding to α_0 into its normal Dynkin diagram and n_{0i}'s, m_{0i}'s and g_{0i}'s are defined in the same way as other n_{ij}'s, m_{ij}'s and g_{ij}'s. Then we have the analogue theorem to Theorem $A.2$.

Theorem A.3. *Suppose R is an irreducible reduced root system, its extended Dynkin diagram is isomorphic to one of the following types of diagrams:*

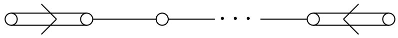
$\tilde{A}_n (n \geq 2)$



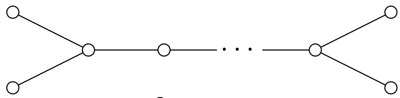
$\tilde{B}_n (n \geq 3)$



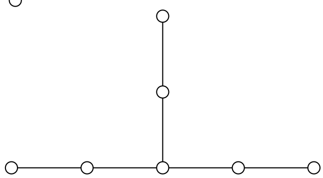
$\tilde{C}_n (n \geq 3)$



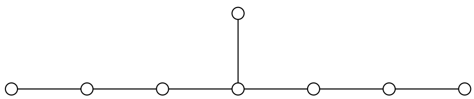
$\tilde{D}_n (n \geq 4)$



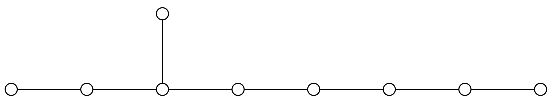
$\tilde{E}_6$



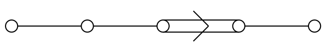
$\tilde{E}_7$



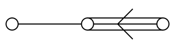
$\tilde{E}_8$



$\tilde{F}_4$



$\tilde{G}_2$



No two of these diagrams are isomorphic.

In order to determine whether these diagrams are positive definite or not, we need the very important following lemma to give an inductive way to compute the determinant of their Gram matrix. This technique will also be used to compute the determinant of the Hermitian form in Chapter 3.

Lemma A.4. *Suppose a (extended) Dynkin diagram with $n = p + q$ nodes is made of two Dynkin subdiagrams, one with p nodes and another one with q nodes. There is only one pair of nodes, each of which from distinct Dynkin subdiagrams, connected by bond(s). Then let's say the last node of the first Dynkin subdiagram is connected to the first node of the second Dynkin subdiagram. Let*

$$G_n = (g_{ij})_{1\leq i,j\leq n}$$

be the Gram matrix of the full Dynkin diagram. Let

$$G_p = (g_{ij})_{1\leq i,j\leq p}, \quad G_q = (g_{ij})_{p+1\leq i,j\leq n}$$

be the Gram matrices of the two Dynkin subdiagrams. Let

$$G_{p-1} = (g_{ij})_{1\leq i,j\leq p-1}, \quad G_{q-1} = (g_{ij})_{p+2\leq i,j\leq n}$$

be the Gram matrices of the two new Dynkin subdiagrams obtained by deleting the last node (resp. the first node) from the first Dynkin subdiagram (resp. the second Dynkin subdiagram) together with all bonds connected to these two nodes. Then we have

$$\det(G_n) = \det(G_p)\det(G_q) - 4\cos^2(\pi/m)\det(G_{p-1})\det(G_{q-1}).$$

Proof. Since the two Dynkin subdiagrams are only connected by the bond(s) between the pth and $(p+1)$th nodes, we have g_{ij}'s vanish for all $1 \leq i \leq p, p+1 \leq j \leq n$ and $p+1 \leq i \leq n, 1 \leq j \leq p$, except for

$$g_{p(p+1)} = g_{(p+1)p} = -2\cos(\pi/m).$$

Hence the formula follows easily from the Leibniz formula

$$\det(G_n) = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\prod_{l=1}^{n} g_{l\sigma(l)}$$

as long as we notice that all summands over $\sigma \in \mathfrak{S}_n$ vanish except for $\sigma \in \mathfrak{S}_p \times \mathfrak{S}_q$ and for $\sigma \in \mathfrak{S}_{p-1} \times (p\ p+1) \times \mathfrak{S}_{q-1}$. □

Using this lemma, we can quickly know that the Dynkin diagrams (resp. extended Dynkin diagrams) for root systems are all positive definite (resp. positive semidefinite).

A.2. Proof of Theorem 3.18

Now we can make use of this technique to compute the determinant of the Hermitian form (3.1) for type $ABCFG$, given by Theorem 3.18. Let's recall how the Hermitian form is given first.

$$h_{ij} = \begin{cases} q_i^{\frac{1}{2}} + q_i^{-\frac{1}{2}} & \text{if} \quad i = j \\ -\, s_{ij} & \text{if} \quad i \neq j \end{cases} \tag{A.1}$$

among which q_i and s_{ij} are given in Section 3.3.

As the completed Dynkin diagram for type A_n is a loop which is somewhat different from other types, we compute this determinant first. But before we proceed to this, we need to look at the determinant of its normal type. Its Hermitian form is given as follows according to $(A.1)$.

$$h_{ij}(A_n) = \begin{cases} 2\cos \pi k & \text{if} \quad i = j \\ -\, e^{-\pi\sqrt{-1}k'} & \text{if} \quad i = j+1 \\ -\, e^{\pi\sqrt{-1}k'} & \text{if} \quad i = j-1 \\ 0 & \text{otherwise} \end{cases}$$

since

$$q^{\frac{1}{2}} = \exp(-\pi\sqrt{-1}k), \quad q'^{\frac{1}{2}} = \exp(-\pi\sqrt{-1}k').$$

Then we have the determinant of the Hermitian form of type A_n.

Lemma A.5. *If the Hermitian form of type A_n is given as above, then we have*

$$\det(h(A_n)) = \begin{cases} 1 + \displaystyle\sum_{l=1}^{m} 2\cos 2l\pi k & \textit{if} \quad n = 2m \\ \displaystyle\sum_{l=1}^{m} 2\cos(2l-1)\pi k & \textit{if} \quad n = 2m-1 \end{cases}$$

Proof. Using the technique in Lemma $A.4$, we immediately have this inductive formula

$$\det(h(A_n)) = 2\cos \pi k \det(h(A_{n-1})) - \det(h(A_{n-2})).$$

Then the result follows by simple calculation. □

With this on hand, we can proceed to compute the determinant of the Hermitian form of the affine type $\tilde{A}_n$.

Theorem A.6. *If the Hermitian form of type $\tilde{A}_n$ is given as in $(A.1)$, then we have*

$$\det(h(\tilde{A}_n)) = -4 \sin \frac{(n+1)\pi(k+k')}{2} \sin \frac{(n+1)\pi(k-k')}{2}.$$

PROOF. Since the completed Dynkin diagram of $\tilde{A}_n$ will not be split into two independent subdiagrams by cutting off only one bond, we can not apply the lemma directly. So we have to compute it step by step.

$$\begin{aligned}
\det(h(\tilde{A}_n)) =& 2\cos\pi k \det(h(A_n)) \\
& - (-e^{-\pi\sqrt{-1}k'})\Big((-e^{\pi\sqrt{-1}k'})\det(h(A_n)) \\
& \qquad + (-1)^{1+n}(-e^{-\pi\sqrt{-1}k'})(-e^{\pi\sqrt{-1}k'})^{n-1}\Big) \\
& + (-1)^{n+2}(-e^{\pi\sqrt{-1}k'})\Big((-1)^{n+1}(-e^{-\pi\sqrt{-1}k'})\det(h(A_n)) \\
& \qquad + (-e^{\pi\sqrt{-1}k'})(-e^{\pi\sqrt{-1}k'})^{n-1}\Big) \\
=& 2\cos\pi k \det(h(A_n)) - \det(h(A_{n-1})) - e^{-\pi\sqrt{-1}(n+1)k'} \\
& - \det(h(A_{n-1})) - e^{\pi\sqrt{-1}(n+1)k'} \\
=& \det(h(A_{n+1})) - \det(h(A_{n-1})) - 2\cos(n+1)\pi k' \\
=& 2\cos(n+1)\pi k - 2\cos(n+1)\pi k' \\
=& -4\sin\frac{(n+1)\pi(k+k')}{2}\sin\frac{(n+1)\pi(k-k')}{2}.
\end{aligned}$$

□

Now we shall compute the determinant of the Hermitian form of affine type $\tilde{B}_n$. But we have to compute the determinant of the Hermitian form of type D_n first.

Lemma A.7. *If the Hermitian form of type $\tilde{D}_n$ is given as in (A.1), then we have*

$$\det(h(D_n)) = 4\cos\pi k\cos(n-1)\pi k.$$

PROOF. We derive it inductively. We can compute

$$\det(h(D_3)) = 4\cos\pi k\cos 2\pi k$$

directly. We cut off the bond connecting the first node and second node, then by Lemma A.4, we have

$$\begin{aligned}
\det(h(D_{n+1})) &= 2\cos\pi k\cdot\det(h(D_n)) - \det(h(D_{n-1})) \\
&= 2\cos\pi k\cdot 4\cos\pi k\cos(n-1)\pi k - 4\cos\pi k\cos(n-2)\pi k \\
&= 4\cos\pi k\cos n\pi k.
\end{aligned}$$

□

Theorem A.8. *If the Hermitian form of type $\tilde{B}_n$ is given as in (A.1), then we have*

$$\det(h(\tilde{B}_n)) = -4\sin\pi((n-2)k + k')\sin 2\pi k.$$

Proof. Cutting off the bond between the $(n-1)$-th node and the n-th node and using Lemma $A.4$, we have

$$\begin{aligned}\det(h(\tilde{B}_n) =& 2\cos\pi k' \cdot \det(h(D_n)) - 2\cos\pi(k-k') \cdot \det(h(D_{n-1}))\\ =& 2\cos\pi k' \cdot 4\cos\pi k\cos(n-1)\pi k\\ & - 2\cos\pi(k-k') \cdot 4\cos\pi k\cos(n-2)\pi k\\ =& -4\sin\pi((n-2)k+k')\sin 2\pi k.\end{aligned}$$

□

The determinant of the Hermitian form (3.1) for type CFG could also be deduced in the same way. Therefore, we obtain the result in Theorem 3.18.

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Index

Samenvatting

In deze scriptie bestuderen we de meetkundige structuren op complementen van torische schikkingen. Dit fenomeen kan ook beschouwd worden als de meetkunde van het gekwantiseerde periodische Calogero-Moser-systeem geassocieerd aan een wortelsysteem.

Het klassieke Calogero-Moser-systeem beschrijft een eindig aantal identieke puntdeeltjes op de reële lijn die onder de invloed zijn van een omgekeerd kwadratische potentiaal. De relatieve posities van deze punten worden geparametriseerd door het complement van een hypervlakschikking van type A_n in het quotiënt van een $(n+1)$-dimensionale vectorruimte naar zijn hoofddiagonaal. Deze scriptie generaliseert dit fenomeen naar zowel een willekeurig wortelsysteem, als naar het periodieke geval.

Als eerste construeren we een projectieve structuur op het complement van een torische schikking. Het idee hier achter is dat we een projectieve structuur op een complexe variëteit M kunnen schrijven in termen van een affiene structuur op $M \times \mathbb{C}^\times$. Op zijn beurt is het welbekend dat een affiene structuur op een complexe variëteit gegeven wordt door een torsievrije en vlakke connectie op zijn (co)raakbundel, en omgekeerd. Het construeren van een projectieve structuur op M is dus equivalent met het produceren van een torsievrije en vlakke connectie op $M \times \mathbb{C}^\times$.

Onze opzet is als volgt. We beginnen met een algebraïsche torus $H := \mathrm{Hom}(Q, \mathbb{C}^\times)$ gegeven door een wortelrooster $Q := \mathbb{Z}R$, waar R een gereduceerd, irreducibel wortelsysteem is. Noteer de Weyl-groep van R met W. Verder is ons ook een torische schikking gegeven, geassocieerd aan het wortelsysteem R. Dit is een eindige verzameling van hypertori, elk gedefinieerd door $H_\alpha := \{h \in H \mid e^\alpha(h) = 1\}$ waar e^α een karakter van H is. We schrijven H° voor het complement van de vereniging van deze hypertori.

Geïnspireerd door het speciale hypergeometrische systeem geconstrueerd door Heckman en Opdam, beschouwen we nu een familie van connecties $\tilde{\nabla}^\kappa = \tilde{\nabla}^0 + \tilde{\Omega}^\kappa$ op de coraakbundel van $H^\circ \times \mathbb{C}^\times$. Hier is κ een W-invariante multipliciteitsparameter voor R, gedefinieerd door $\kappa := (k_\alpha)_{\alpha \in R} \in \mathbb{C}^R$, $\tilde{\nabla}^0$ staat voor de (vlakke) translatie-invariante connectie op $H \times \mathbb{C}^{\times}$, en $\tilde{\Omega}^{\kappa}$ is de volgende connectievorm:

$$\tilde{\Omega}^{\kappa} := \frac{1}{2}\sum_{\alpha>0} k_{\alpha}\frac{e^{\alpha}+1}{e^{\alpha}-1} d\alpha \otimes d\alpha \otimes \partial_{\alpha^{\vee}} + (B^{\kappa})^{*} + c^{\kappa}\sum_{\alpha>0} d\alpha \otimes d\alpha \otimes t\frac{\partial}{\partial t}$$
$$- \sum_{\alpha_i \in \mathfrak{B}} d\alpha_i \otimes \frac{dt}{t} \otimes \partial_{p_i} - \frac{dt}{t} \otimes \frac{dt}{t} \otimes t\frac{\partial}{\partial t} - \sum_{\alpha_i \in \mathfrak{B}} \frac{dt}{t} \otimes d\alpha_i \otimes \partial_{p_i}.$$

Hier staat t voor de coördinaat op $\mathbb{C}^{\times}$, c^{κ} is een constante voor iedere κ, B^{κ} staat voor een gegeven translatie-invariant tensorveld op H of $H \times \mathbb{C}^{\times}$ en $\mathfrak{B}$ is een fundamenteel systeem voor R.

Het is duidelijk dat $\tilde{\nabla}^{\kappa}$ torsievrij is, maar om in te zien dat deze vlak is kost meer moeite. Om dit te na te gaan passen we een vlakheidscriterium toe dat is opgezet door Looijenga, of eerder door Kohno. Dit criterium vereist dat we eerst $H^{\circ} \times \mathbb{C}^{\times}$ compactificeren en vervolgens de residuen van $\tilde{\Omega}^{\kappa}$ berekenen langs de toegevoegde spiegels en randdivisoren. Toepassen van het criterium op onze situatie levert condities op voor het vlak zijn van $\tilde{\nabla}^{\kappa}$, zodat tenslotte een W-invariante projectieve structuur op H° geconstrueerd kan worden in termen van $\tilde{\nabla}^{\kappa}$.

Vervolgens laten we zien dat het complement H° van de torische schikking een hyperbolische structuur toelaat wanneer de multipliciteitsparameter κ in een bepaald gebied ligt. Deze conditie betekent namelijk dat het beeld van H° onder de projectieve evaluatieafbeelding in een complexe bal belandt. Het idee is dat we eerst de monodromierepresentatie van het systeem met de reflectierepresentatie identificeren en dus voor iedere κ een Hermitische vorm h op het beeld van de evaluatieafbeelding kunnen definiëren. Nu kunnen we het relevante 'hyperbolische' gebied voor κ vinden door de determinant van h te berekenen. We bewijzen dan dat zijn duale Hermitische vorm h^{*} groter dan 0 is (of equivalent, dat $h < 0$), waarmee het gewenste resultaat volgt.

Laten we dit iets verder uitleggen. We berekenen eerst de eigenwaarden van de residu-endomorfismen van $\tilde{\nabla}^{\kappa}$ langs spiegels en randdivisoren respectievelijk en een verrassend feit is dat elk residu-endomorfisme hoogstens twee eigenwaarden heeft, of dit nu langs een spiegel of een randdivisor is. Dit vertelt ons in het bijzonder hoe het lokale gedrag van de evaluatieafbeelding er uit ziet voor de affiene structuur rond deze divisoren.

Daarna construeren we de reflectierepresentatie van de zogenaamde affiene Artin-groep $\mathrm{Art}(M)$, waar M de affiene Coxeter-matrix geassocieerd aan het affiene wortelsysteem van R is. Door gebruik te maken van een stelling van Brieskorn kan de uitgebreide affiene Artin-groep $\mathrm{Art}'(M) := \mathrm{Art}(M) \rtimes (P^{\vee}/Q^{\vee})$ worden geïdentificeerd met de fundamentele groep van de orbifold $W\backslash H^{\circ}$. Als gevolg kan overeenkomstig de reflectierepresentatie met de monodromierepresentatie van het systeem geïdentificeerd worden.

Verder definiëren we een Hermitische vorm h op de bijbehorende doelruimte vanuit het oogpunt van de reflectierepresentatie, zodat we het hyperbolische gebied van het systeem kunnen verkrijgen door de determinant van h te onderzoeken. In onze situatie kunnen we de evaluatieafbeelding rond de subreguliere punten uitschrijven in de vorm van lokale coördinaten in termen van de eigenwaarden van de residu-endomorfismen, waar we deze eigenwaarden zien als lokale exponenten. Hier bedoelen we met subreguliere punten de punten die liggen in precies één spiegel of randdivisor. Na deze voorbereiding kunnen we uiteindelijk bewijzen dat de duale Hermitische vorm h^* groter dan 0 is wanneer κ in het hyperbolische gebied ligt, zodat de Γ-overdekking van $W\backslash H^\circ$ de structuur van een complexe bal toelaat. Hier staat Γ voor de projectieve monodromiegroep.

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Curriculum Vitae

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